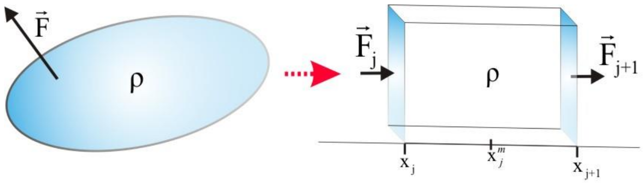
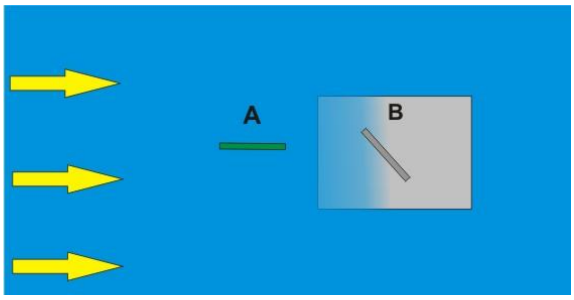
The advection-diffusion heat equation:

$$\boxed{\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}} \xrightarrow{+\text{ Advection}} \frac{\partial T}{\partial t} + \underbrace{\frac{\partial VT}{\partial x}}_{\text{Advection term}} = \chi \frac{\partial^2 T}{\partial x^2},$$

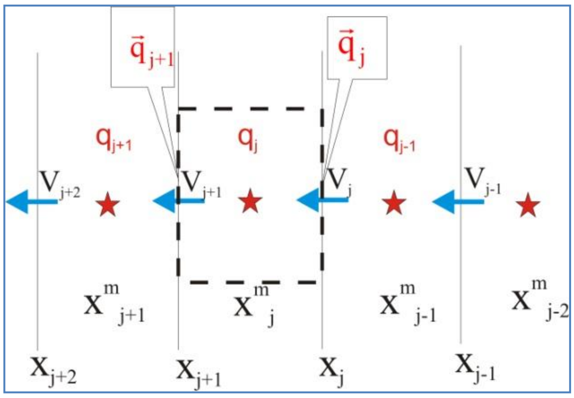


where V= f(x), for example: V = 1.

Upwind – discretization: the idea here is that the matter/energy in a test volume V is influenced mainly by the flux coming from the up-stream direction rather than from the down-stream.



The upwind diskretization procedure:



$$\frac{\partial VT}{\partial x} \xrightarrow{Upwind} \frac{V_j \cdot \vec{T}_j - V_{j+1} \cdot \vec{T}_{j+1}}{\Delta x_j},$$

$$where \quad \vec{T}_j = \begin{cases} T_{j-1} & \text{if } V_j < 0 \\ T_j & \text{if } V_j \geq 0 \end{cases}$$

$$\Leftrightarrow \vec{T}_j = \alpha_j T_{j-1} + (1-\alpha_j) T_j$$

$$\frac{\partial VT}{\partial x} \xrightarrow{Upwind} \frac{V_j \cdot [\alpha_j T_{j-1} + (1-\alpha_j)T_j] - V_{j+1} \cdot [\alpha_{j+1} T_j + (1-\alpha_{j+1}) T_{j+1}]}{\Delta x_j}$$

$$= \left[\frac{\alpha_j V_j}{\Delta x_j}\right] T_{j-1} + \left[\frac{(1-\alpha_j)V_j}{\Delta x_j} - \frac{\alpha_{j+1} V_{j+1}}{\Delta x_j}\right] T_j - \left[\frac{(1-\alpha_{j+1})V_{j+1}}{\Delta x_j}\right] T_{j+1}$$

CALL SUPWIND(T,V,α):

```
IF(V_j<0) THEN
      α(j) = 1
   ELSE
      α(j) = 0
   ENDIF
```

In the explicit case, the diffusion and advection operators are evaluated, using the values from the old time level. The procedure runs as follows:

Explicit calculation:

$$\frac{T_j^{n+1}-T_j^n}{\delta t} + \frac{V_j^n \cdot \vec{T}_j^n - V_{j+1}^n \cdot \vec{T}_{j+1}^n}{\Delta x_j} = \frac{1}{\Delta x}\left[\frac{T_{j+1}^n - T_j^n}{\Delta x} - \frac{T_j^n - T_{j-1}^n}{\Delta x}\right]$$

$$\frac{T_j^{n+1}-T_j^n}{\delta t} = -\left[\frac{V_j^n \cdot \mathrm{TUPW}_j^n - V_{j+1}^n \cdot \mathrm{TUPW}_{j+1}^n}{\Delta x_j}\right] + \frac{1}{\Delta x}\left[\frac{T_{j+1}^n - T_j^n}{\Delta x} - \frac{T_j^n - T_{j-1}^n}{\Delta x}\right]$$

$$\Leftrightarrow T_j^{n+1} = sT_{j-1}^n + (1-2s)T_j^n + sT_{j+1}^n - \left[\beta_j \cdot \mathrm{TUPW}_j^n - \beta_{j+1} \cdot \mathrm{TUPW}_{j+1}^n\right],$$

where $\begin{cases} s = \dfrac{\delta t}{\Delta x^2} \\ \beta_j = \dfrac{\delta t \cdot V_j}{\Delta x} \end{cases}$ and UPWIND(T,V,TUP):

```
IF(V_j <0) THEN
    TUP(j) = T^n_{j-1}
ELSE
    TUP(j) =T^n_j
ENDIF
```

==

$Q1$: Construct an explicit solver for solving the following advection-diffusion equation:

$$\frac{\partial T}{\partial t} + \frac{\partial UT}{\partial x} = \chi \frac{\partial^2 T}{\partial x^2},$$ in the interval $[0 \le x \le 1]$ subject to the ICs:

$$T(t=0, x) = \begin{cases} 1 & x \le 0.5 \\ 0 & \text{else} \end{cases}, \quad \text{BCs: } T = \begin{cases} 1 & x=0 \\ 0 & x=1 \end{cases},$$ where U = const.= 1.

Use Δx=0.01 to show the resulting profiles (overplotted) of T after time=0.1, 0.25, 0.5, 0.75 and 1.0 and using χ =1.0, 0.5, 0.1, 0.01 and 0.001. Use the upwind discretization and compare it with the central difference scheme.

➢ Linear systems of equations –(LSEs)

The system of equations:

$$\begin{array}{ll} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 & \cdots(1) \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 & \cdots(2) \\ \quad\vdots \qquad \vdots \qquad \vdots & \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_K & \cdots(n) \end{array}$$

is said to be **linear**, if **a_{jk}** are constant coefficients and do not depend on the solution itself. Xs are unknowns and the bs are known quantities.

Example-1:

- The system of equations: $\boxed{\begin{array}{l} -x_1 + 2x_2 - x_3 = 0.6 \\ x_1 + 1x_2 - 2x_3 = 0.8 \\ -2x_1 + 2x_2 - 3x_3 = 1.2 \end{array}}$ is linear.

- But the following systems of equations:
$\boxed{\begin{array}{l} -x_1^{\boxed{2}} + 2x_2 - x_3 = 0.6 \\ x_1 + 1x_2 - 2x_3 = 0.8 \\ -2x_1 + 2x_2 - 3x_3^{\boxed{3/2}} = 1.2 \end{array}}$ $\boxed{\begin{array}{l} -x_1 + \boxed{2x_1}x_2 - x_3 = 0.6 \\ x_1 + 1x_2 - 2x_2x_3 = 0.8 \\ -4x_1\boxed{x_3} + 2x_2 - 3x_3 = 1.2 \end{array}}$
are <u>nonlinear</u>.

Example-2:

$$\boxed{\begin{array}{l} 2x_1 - 1x_2 = 1 \\ x_1 + 2x_2 = -2 \end{array}};\ 2\times \text{Eq2} - \text{Eq1}$$

$$\Rightarrow \boxed{\begin{array}{l} 2x_1 - 1x_2 = 1 \\ 0x_1 + 5x_2 = -5 \end{array}} \Rightarrow \begin{cases} x_2 = -1 \\ x_1 = 0 \end{cases}$$

This elimination method is due to Gauss.

Generalization of the Gauss elimination method:

Assume we are given the following set of linear equations:

$$\boxed{\begin{array}{ll}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 & \cdots(1)\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 & \cdots(2)\\
\qquad\vdots \qquad\qquad \vdots \qquad\qquad \vdots & \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_K & \cdots(n)
\end{array}}$$

We first divide equation (1) by the coefficient of x_1. Equation (1) now reads:

$$x_1 + \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3 + \ldots + \frac{a_{1n}}{a_{11}}x_n = \frac{b_1}{a_{11}} \qquad \ldots (1)$$

Then we multiply equation (1) with the coefficient of x_1 from equation (2):

$$a_{21}x_1 + a_{21}\frac{a_{12}}{a_{11}}x_2 + a_{21}\frac{a_{13}}{a_{11}}x_3 + \ldots + a_{21}\frac{a_{1n}}{a_{11}}x_n = a_{21}\frac{b_1}{a_{11}} \qquad \ldots (1)$$

and subtract it from equation (2):

$$(a_{21} - a_{21})x_1 + \underbrace{\left(a_{22} - a_{21}\frac{a_{12}}{a_{11}}\right)}_{a'_{22}}x_2 + \underbrace{\left(a_{23} - a_{21}\frac{a_{13}}{a_{11}}\right)}_{a'_{23}}x_3 + \ldots + \underbrace{\left(a_{2n} - a_{21}\frac{a_{1n}}{a_{11}}\right)}_{a'_{2n}}x_n = \underbrace{b_2 - a_{21}\frac{b_1}{a_{11}}}_{b'_2}$$

We may re-write this equation in the following compact form:

$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2 \qquad \ldots (2)$$

Similarly, we adopt this procedure to eliminate all the coefficients $\boxed{a'_{1\to n,2}}$, so to get the system in the form:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \quad \ldots (1)\\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \quad \ldots (2)\\
a'_{32}x_2 + a'_{33}x_3 + \ldots + a'_{3n}x_n &= b'_3 \quad \ldots (3)\\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots\\
a'_{n2}x_2 + a'_{n3}x_3 + \ldots + a'_{nn}x_n &= b'_n \quad \ldots (n)
\end{aligned}
$$

We now focus our attention on the Equations (**3→n**) and repeat the above-described procedure to eliminate

The coefficients $a''_{3\to n,3}$ by suitable multiplication and subtraction Eq. (2) from them. The same procedure is employed then for Eq (4) and then for Eq.(5) until Eq. (n) is recovered, so to end up with the following triangular form:

$$
\boxed{
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \quad \ldots (1)\\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \quad \ldots (2)\\
a''_{33}x_3 + \ldots + a''_{3n}x_n &= b''_3 \quad \ldots (3)\\
\vdots \qquad \vdots \qquad &\qquad \vdots\\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1)} \quad \ldots (n)
\end{aligned}
}
$$

But the last equation, i.e., Eq. (n), has a single unknown only, so that it can be solved:

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Where $\square^{n-1}$ denote the value of the coefficient after performing $\boxed{\text{n-1}}$ algebraic manupulation.

Once we obtained the value of **x_n**, we can then use it to backward-substitute it

in Eq. (n-1) , which is in turn used for backward substitution in Eq. (n-2) and so on as follows:

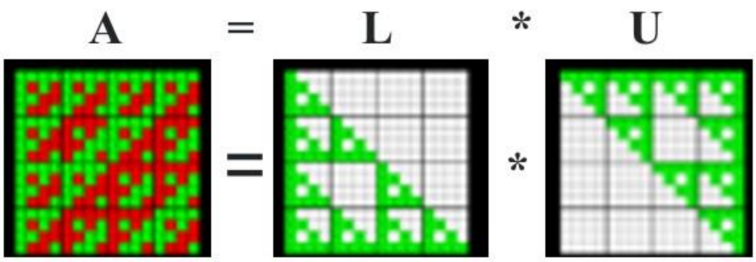
$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}}$$

Computational costs (Gauss Elimination):

- Forward elimination ~ $\mathbf{n^3/3}$
- Backward /forward substitution ~ $\mathbf{n^2}$

Total operations: ~ $\mathbf{n^3}$

LU- decomposition:



Since the inversion of a triangular matrix is actually a straightforward procedure (***just backward substitution, and therefore scales as n^2***), it is suggested to examine the possibility of decomposing the Matrix in such a manner, that it is the product of two triangular matrices:

Example:

Consider the matrix A:

$$A=\begin{pmatrix} 2 & 3 & 1 & 5 \\ 6 & 13 & 5 & 19 \\ 2 & 19 & 10 & 23 \\ 4 & 10 & 11 & 31 \end{pmatrix} = \begin{pmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ l_{31} & l_{32} & l_{33} & \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix} \bullet \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ & u_{22} & u_{23} & u_{24} \\ & & u_{33} & u_{34} \\ & & & u_{44} \end{pmatrix}$$

$\Rightarrow$The equations to be solved:

$l_{11} \bullet u_{11} = 2$

$l_{11} \bullet u_{12} = 3$ and so on $\cdots$

Q2: complete the set of equations to be solved and prove that:

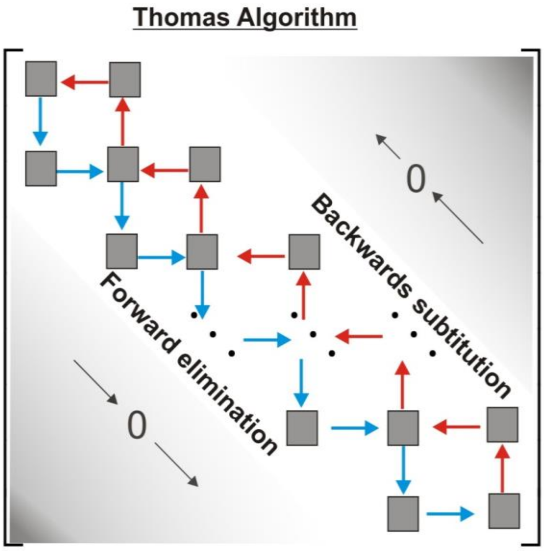
$$L=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 2 & 1 & 7 & 1 \end{pmatrix},\ U=\begin{pmatrix} 2 & 3 & 1 & 5 \\ 0 & 4 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Computational costs(LU): could be smaller or even larger than that of Gauss elimination ($\mathbf{n^3}$), depending on the sparsity of the matrix A.

Thomas Algorithm:

Given is the following linear system of equations:

$$\begin{cases} +b_1x_1 - c_1x_2 = d_1 \\ -a_2x_2+b_2x_2 - c_2x_3 = d_2 \\ \ddots \\ \ddots \\ -a_{n-1}x_{n-1}+b_nx_n = d_n \end{cases}$$



This set of equations may be solved as follows:

1. Eliminate $\mathbf{x_1}$ from Eq.1 and substitute in Eq. 2
2. Eliminate $\mathbf{x_2}$ from Eq.2 and substitute in Eq. 3
3. Apply this **forward elimination** until Eq. (n) is recovered and then solve it for $\mathbf{x_n}$.
4. Perform the **backward substitution** until Eq.1 is reached.

The tri-diagonal non-pivoting elimination method is stable if:

1. a_j, b_j and $c_j > 0$ ⇔ diagonal elements must be positive, the off-diagonal are negative.
2. $b_j > a_j + c_j$ ⇔ row-diagonally dominant
3. $b_j > a_{j+1} + c_{j-1}$ ⇔ column-diagonally dominant

Q1/L7&8: Solve the following set of linear equations, using the Thomas Algorithm:

$$\begin{aligned} 2x_1 - x_2 &= 0.2 \\ -x_1 + 2x_2 - x_3 &= 0.6 \\ -x_2 + 2x_3 - x_4 &= 0.8 \\ -x_3 + 2x_4 - x_5 &= 1.2 \\ -2x_4 + 2x_5 &= 1.2 \end{aligned}$$

Implicit solution procedure

The heat equation

$$\boxed{\frac{\partial T}{\partial t} = \chi \frac{\partial^2 T}{\partial x^2}}.$$

An implicit discretization method yields:

$$\frac{T_j^{n+1} - T_j^n}{\delta t} = \frac{\chi}{\Delta \mathrm{x}}\left[\frac{T_{j+1}^{n+1} - T_j^{n+1}}{\Delta \mathrm{x}} - \frac{T_j^{n+1} - T_{j-1}^{n+1}}{\Delta \mathrm{x}}\right]$$

$$\Leftrightarrow T_j^{n+1} = T_j^n + \left[sT_{j-1}^{n+1} - 2sT_j^{n+1} + sT_{j+1}^{n+1}\right], \quad \text{where } s = \frac{\delta t \times \chi}{\Delta \mathrm{x}^2}$$

or equivalently:

$$\left[-sT_{j-1}^{n+1} + (1+2s)T_j^{n+1} - sT_{j+1}^{n+1}\right] = T_j^n$$

Let s = 1, then

$$\left[-T_{j-1}^{n+1} + 3T_j^{n+1} - T_{j+1}^{n+1}\right] = T_j^n$$

$$j = 2$$

$$\mapsto \quad 3T_2^{n+1} - T_3^{n+1} = T_2^n + T_1^{BC}$$

$$j = 3$$

$$\mapsto \quad -T_2^{n+1} + 3T_3^{n+1} - T_4^{n+1} = T_3^n$$

...

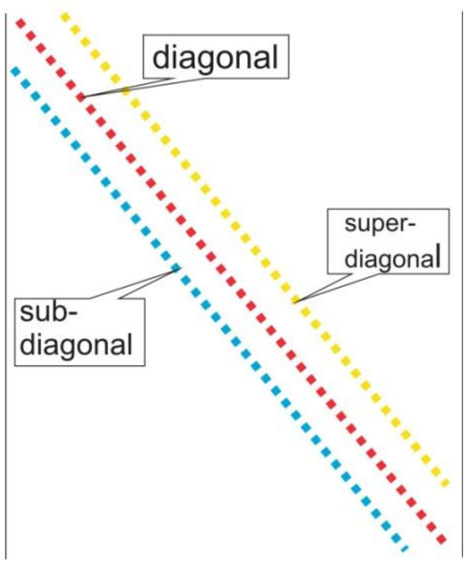
$$j = J$$

$$\mapsto \quad -T_{J-1}^{n+1} + 3T_J^{n+1} = T_J^n + T_{J+1}^{BC}$$

$$\begin{pmatrix} 3 & -1 & & \\ -1 & 3 & -1 & \\ & & \ddots & -1 \\ & & -1 & 3 \end{pmatrix} \begin{pmatrix} T_2^{n+1} \\ T_3^{n+1} \\ T_j^{n+1} \\ T_J^{n+1} \end{pmatrix} = \begin{pmatrix} T_2^n + T_1^n \\ T_3^n \\ T_j^n \\ T_J^n + T_{J+1}^n \end{pmatrix}$$

$$\Leftrightarrow \quad \overrightarrow{\overrightarrow{\mathrm{A}}} \quad \vec{T}^{n+1} = \overrightarrow{\mathrm{RHS}}^{\mathrm{n}}$$

Tridiagonal matrix



The Tri-diagonal solver: GTSL

CALL GTSL(Nel,Sub,Diag,Super,RHS,INFO)
where
Nel = total number of diagonals
Sub = sub-diagonal entries
Diag = diagonal entries
Super = super-diagonal entries
RHS = the entries on the right hand side
INFO = 0, if all entries are defined
k, if entry number k is not appropriately defined

Q3: Use the GTSLs solver to solve the following system of linear equations: $\left[-T_{j-1} + 3T_j - T_{j+1}\right] = T_j^{old}$,

where $T_j^{old} = 1 + e^{-\left(\frac{j-50}{100}\right)^2}$, $T_1 = T_{100} = 1$ and for $j = 1, 2, \ldots 100$.

The Crank-Nicholson Method:

$$\frac{T_j^{n+1} - T_j^n}{\delta t} = \left\langle \frac{1}{2} \right\rangle \frac{\chi}{\Delta x}\left[\frac{T_{j+1}^n - T_j^n}{\Delta x} - \frac{T_j^n - T_{j-1}^n}{\Delta x}\right]^{old}$$

$$+ \left\langle \frac{1}{2} \right\rangle \frac{\chi}{\Delta x}\left[\frac{T_{j+1}^{n+1} - T_j^{n+1}}{\Delta x} - \frac{T_j^{n+1} - T_{j-1}^{n+1}}{\Delta x}\right]^{new}$$

$$\Leftrightarrow T_j^{n+1} = sT_{j-1}^n + (1-2s)T_j^n + sT_{j+1}^n$$

$$+ sT_{j-1}^{n+1} - 2sT_j^{n+1} + sT_{j+1}^{n+1}, \quad \text{where} \quad s = \frac{\delta t \times \chi}{2 \times \Delta x^2}.$$

$$\Leftrightarrow -sT_{j-1}^{n+1} + (1+2s)T_j^{n+1} - sT_{j+1}^{n+1} = \underbrace{sT_{j-1}^n + (1-2s)T_j^n + sT_{j+1}^n}_{\text{RHS}}$$

$$
\Leftrightarrow \begin{vmatrix}
1+2s & -s & & & \\
-s & 1+2s & -s & & \\
 & & \ddots & & \\
 & & -s & 1+2s & -s \\
 & & & -s & 1+2s
\end{vmatrix}
\begin{bmatrix} T_2^{n+1} \\ T_3^{n+1} \\ \vdots \\ T_{J-1}^{n+1} \\ T_{J-1}^{n+1} \end{bmatrix}
=
\begin{bmatrix} \mathrm{RHS}_2^n \\ \mathrm{RHS}_3^n \\ \vdots \\ \mathrm{RHS}_{J-1}^n \\ \mathrm{RHS}_{J-1}^n \end{bmatrix}
$$

$$
\Leftrightarrow \begin{pmatrix}
(1+2s) & -s & & \\
-s & (1+2s) & -s & \\
 & & \ddots & -s \\
 & & -s & (1+2s)
\end{pmatrix}
\begin{bmatrix} T_2^{n+1} \\ T_3^{n+1} \\ T_j^{n+1} \\ \vdots \\ T_{J-1}^{n+1} \end{bmatrix}
=
\begin{bmatrix} T_2^n + T_1 \\ T_3^n \\ T_j^n \\ \vdots \\ T_J^n + T_J \end{bmatrix}
$$

The advection-diffusion heat equation: implicit solution procedure

In the implicit case, the diffusion and advection operators are evaluated, using the values from the **NEW** time level. The procedure runs as follows:

Implicit calculation:

$$\frac{T_j^{n+1}-T_j^n}{\delta t} + \frac{V_j^n \cdot \vec{T}_j^{\,n+1} - V_{j+1}^n \cdot \vec{T}_{j+1}^{\,n+1}}{\Delta x_j} = \frac{1}{\Delta x_j}\left[\frac{T_{j-1}^{n+1}-T_j^{n+1}}{\Delta x_j^m} - \frac{T_j^{n+1}-T_{j+1}^{n+1}}{\Delta x_{j+1}^m}\right]$$

$$\frac{T_j^{n+1}-T_j^n}{\delta t} + \frac{V_j^n \cdot [\alpha_j T_{j-1}^{n+1} + (1-\alpha_j)T_j^{n+1}] - V_{j+1}^n \cdot [\alpha_{j+1} T_j^{n+1} + (1-\alpha_{j+1})T_{j+1}^{n+1}]}{\Delta x_j}$$

$$= \frac{1}{\Delta x_j}\left[\frac{T_{j+1}^{n+1}-T_j^{n+1}}{\Delta x_j^m} - \frac{T_j^{n+1}-T_{j-1}^{n+1}}{\Delta x_{j+1}^m}\right]$$

$$\frac{T_j^{n+1}-T_j^n}{\delta t} + \left[\frac{\alpha_j V_j^n}{\Delta x_j}\right] T_{j-1}^{n+1} + \left[\frac{(1-\alpha_j)V_j^n}{\Delta x_j} - \frac{\alpha_{j+1} V_{j+1}^n}{\Delta x_j}\right] T_j^{n+1} + \left[\frac{(1-\alpha_{j+1})V_{j+1}^n}{\Delta x_j}\right] T_{j+1}^{n+1}$$

$$= \left[\frac{T_{j+1}^{n+1}-T_j^{n+1}}{\Delta x_j \Delta x_j^m} - \frac{T_j^{n+1}-T_{j-1}^{n+1}}{\Delta x_j \Delta x_{j+1}^m}\right]$$

$$\Leftrightarrow \qquad \boxed{\underline{S}_j \cdot T_{j-1}^{n+1} + D_j \cdot T_{j-1}^{n+1} + \overline{S}_j \cdot T_{j-1}^{n+1} = T_j^n}$$

Q4-1: Compute: $\underline{S}_j$, D_j, $\overline{S}_j$.

$Q4-2$: Prove that upwind-discretization boosts the diagonal dominance of the coefficient matrix.

We may combine explicit and implicit time stepping to generate the damped CN-method as follows:

Crank-Nicholson method for advection-diffusion equations:

$$\frac{T_j^{n+1}-T_j^n}{\delta t} + \alpha_{CN}\left[\frac{V_j^n\cdot\vec{T}_j^{n+1} - V_{j+1}^n\cdot\vec{T}_{j+1}^{n+1}}{\Delta x_j}\right] + (1-\alpha_{CN})\left[\frac{V_j^n\cdot\vec{T}_j^{n} - V_{j+1}^n\cdot\vec{T}_{j+1}^{n}}{\Delta x_j}\right]$$

$$= \alpha_{CN}\frac{1}{\Delta x_j}\left[\frac{T_{j-1}^{n+1}-T_j^{n+1}}{\Delta x_j^m} - \frac{T_j^{n+1}-T_{j+1}^{n+1}}{\Delta x_{j+1}^m}\right]$$

$$+ (1-\alpha_{CN})\frac{1}{\Delta x_j}\left[\frac{T_{j-1}^{n}-T_j^{n}}{\Delta x_j^m} - \frac{T_j^{n}-T_{j+1}^{n}}{\Delta x_{j+1}^m}\right]$$

We define the parameter α_{CN}, to be $0 \le \alpha_{CN} \le 1$.

$\Leftrightarrow$

$$\frac{T_j^{n+1}-T_j^n}{\delta t} + \alpha_{CN}\left\{\left[\frac{V_j^n\cdot\vec{T}_j^{n+1} - V_{j+1}^n\cdot\vec{T}_{j+1}^{n+1}}{\Delta x_j}\right] - \frac{1}{\Delta x_j}\left[\frac{T_{j-1}^{n+1}-T_j^{n+1}}{\Delta x_j^m} - \frac{T_j^{n+1}-T_{j+1}^{n+1}}{\Delta x_{j+1}^m}\right]\right\}$$

$$= (1-\alpha_{CN})\left\{\left[\frac{V_j^n\cdot\vec{T}_j^{n} - V_{j+1}^n\cdot\vec{T}_{j+1}^{n}}{\Delta x_j}\right] + \frac{1}{\Delta x_j}\left[\frac{T_{j-1}^{n}-T_j^{n}}{\Delta x_j^m} - \frac{T_j^{n}-T_{j+1}^{n}}{\Delta x_{j+1}^m}\right]\right\}$$

$\Leftrightarrow$

$$\boxed{\begin{aligned}&\underline{S}_j\cdot T_{j-1}^{n+1} + D_j\cdot T_{j-1}^{n+1} + \overline{S}_j\cdot T_{j-1}^{n+1}\\ &= \frac{T_j^n}{\delta t} + (1-\alpha_{CN})\left\{-\left[\frac{V_j^n\cdot\vec{T}_j^{n} - V_{j+1}^n\cdot\vec{T}_{j+1}^{n}}{\Delta x_j}\right] + \frac{1}{\Delta x_j}\left[\frac{T_{j-1}^{n}-T_j^{n}}{\Delta x_j^m} - \frac{T_j^{n}-T_{j+1}^{n}}{\Delta x_{j+1}^m}\right]\right\}\end{aligned}}$$

$Q5$: Construct an implicit tool to solve the following advection-diffusion equation:

$$\frac{\partial T}{\partial t} + \frac{\partial UT}{\partial x} = \chi\frac{\partial^2 T}{\partial x^2},$$ in the interval $[0 \le x \le 1]$ subject to the ICs:

$T(t=0, 0.3 \le x \le 0.5) = 1$, $T(t=0, 0 \le x \le 0.3) = T(t=0, 0.5 \le x \le 1.0) = 0$

and

BCs: $T(0, x=0) = T(t, x=1) = 0$, where U = const.= 1.

Use $\Delta x=0.01$ $\alpha_{CN} = \frac{1}{2}\left(\frac{1+2\delta t}{1+\delta t}\right)$ to show the resulting profiles (overplotted)

of T at t=0.1, 0.25, 0.5, 0.75 and 1.0 for the following χ values:

1.0, 0.5, 0.1, 0.01 and 0.001.