EFFECTIVE TRANSMISSION CONDITIONS FOR REACTION-DIFFUSION PROCESSES IN DOMAINS SEPARATED BY AN INTERFACE

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Abstract. In this paper, we develop multiscale methods appropriate for the homogenization of processes in domains containing thin heterogeneous layers. Our model problem consists of a nonlinear reaction-diffusion system defined in such a domain, and properly scaled in the layer region. Both the period of the heterogeneities and the thickness of the layer are of order \( \varepsilon \). By performing an asymptotic analysis with respect to the scale parameter \( \varepsilon \) we derive an effective model which consists of the reaction-diffusion equations on two domains separated by an interface together with appropriate transmission conditions across this interface. These conditions are determined by solving local problems on the standard periodicity cell in the layer. Our asymptotic analysis is based on weak and strong two-scale convergence results for sequences of functions defined on thin heterogeneous layers. For the derivation of the transmission conditions, we develop a new method based on test functions of boundary layer type.

Key words. nonlinear reaction-diffusion systems, thin heterogeneous layer, homogenization, two-scale convergence, transmission conditions

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1. Introduction. In this paper, we will be concerned with a nonlinear system of reaction-diffusion equations in a domain containing a thin heterogeneous layer. Such problems often occur in applications like, e.g., transdermal diffusion of drugs, diffusion of substances through the epithelial monolayer, and transport of ions through membranes.

We start from a microscopic model defined on a domain containing a thin layer of thickness \( \varepsilon \). The processes are modeled by a system of reaction-diffusion equations properly scaled inside the layer. Our aim is to study the behavior of the solutions of the microscopic equations when the thickness \( \varepsilon \) tends to zero.

In the limit \( \varepsilon \to 0 \) the thin layer reduces to an interface between the two bulk regions. We derive an effective model which consists of a system of reaction-diffusion equations on both sides of this interface together with appropriate transmission conditions for the limit concentrations across the interface.

For the derivation of the limit equations in the bulk regions we use standard compactness results based on classical a priori estimates. However, the thin heterogeneous layer poses additional problems. We have to adapt the concepts of weak and strong two-scale convergences to functions on thin domains with oscillatory and (for simplicity) periodic structure. Due to the arising nonlinearities, it is necessary to prove strong two-scale convergence of the solutions to the \( \varepsilon \)-problems in the thin layer. To this end, we introduce macroscopic and microscopic coordinates and analyze the regularity of the solutions with respect to this pair of coordinates. Whereas the gradients
with respect to the microvariable and the time can be controlled in $L^2$ in a standard way, the control of the dependence on the macrovariable demands a new approach. Using the partial differential equation in the layer and the strong compactness of the traces of the microscopic concentrations on the interfaces between the bulk regions and the thin layer, we are able to derive $L^2$-equicontinuity also with respect to the macrovariable. When these results are combined, a Kolmogorov criterion for strong two-scale convergence in the thin layer is satisfied.

For the derivation of the transmission conditions we introduce a new method which is based on testing the microscopic equations with test functions of boundary layer type. The effective transmission conditions consist of relations for the jump of the concentrations and the normal fluxes across the interface. They are calculated by solving local problems on the standard periodicity cell in the layer. These local problems are coupled with the effective equations in the bulk regions through the boundary values on the upper and lower boundaries of the standard cell.

Our paper is organized as follows: We start with the precise description of the geometry and of the microscopic equations and the formulation of the main results (section 2). Then we show a priori estimates (section 3). In section 4, we present the convergence concepts needed to pass to the limit in the thin layer. Section 5 gives the proofs for the convergence (up to subsequences) of the microscopic solutions in the bulk and in the layer. Here, the most challenging part is to show strong two-scale convergence of the concentration inside the layer. In section 6, we derive the effective model including the macroscopic equations in the bulk regions, the local problem in the layer, and the transmission conditions across the interface separating the bulk regions. In the last section, we show uniqueness for the homogenized solution and thus the convergence of the whole sequence.

Reducing the computational work is one of the main purposes of the homogenization limit. The algorithms for solving the derived transmission problem numerically will be considered in a forthcoming paper.

The limiting methods developed in this paper are crucial in treating the ion transport through membrane channels modeled by the Nernst–Planck equations. This system couples the transport of ions and the electric field in a domain separated by a thin membrane with periodically distributed channels. In the channels, partially fixed charges control the permeability of the membrane. Here again, the derivation of the transmission conditions is the crucial topic. A formulation of the results and a sketch of the steps necessary in the analysis are given in [14], and the full mathematical derivation is presented in the forthcoming paper [15].

2. Statement of the problem and the results.

2.1. Setting of the problem. Let $\varepsilon > 0$ be a sequence of strictly positive numbers tending to zero, with the property that $\frac{1}{\varepsilon} \in \mathbb{N}$, and let $H > 0$ be a fixed real number.

We consider a bounded domain $\Omega = ]0, 1[^n \times ]-H, H[ \subset \mathbb{R}^n$, $n \geq 2$, consisting of three subdomains: the bulk regions $\Omega^+_{\varepsilon}$, $\Omega^-_{\varepsilon}$ and the thin heterogeneous layer $\Omega^M_{\varepsilon}$, separated by the interfaces $S^+_{\varepsilon}$ and $S^-_{\varepsilon}$; see Figure 2.1. Thus we have

$$\Omega = \Omega^+_{\varepsilon} \cup \Omega^-_{\varepsilon} \cup \Omega^M_{\varepsilon} \cup S^+_{\varepsilon} \cup S^-_{\varepsilon},$$

where $\Omega^+_{\varepsilon} = ]0, 1[^n \times ]\varepsilon, H[, \Omega^-_{\varepsilon} = ]0, 1[^n \times ]-H, -\varepsilon[, \Omega^M_{\varepsilon} = ]0, 1[^n \times ]-\varepsilon, \varepsilon[, S^+_{\varepsilon} = ]0, 1[^n \times \{\varepsilon\}, S^-_{\varepsilon} = ]0, 1[^n \times \{-\varepsilon\}$. We denote

$$\Sigma = ]0, 1[^n \times \{0\}.$$
The domain $\Omega$ including the thin heterogeneous layer $\Omega_M^\varepsilon$.

The standard cell $Z = Y \times [-1, 1] = [0, 1]^{n-1} \times [-1, 1]$.

The microscopic structure of the layer $\Omega_M^\varepsilon$ is obtained by the periodic repetition of the standard cell $Z$ (see Figure 2.2), scaled with $\varepsilon$. Here

$$Z = Y \times [-1, 1] = [0, 1]^{n-1} \times [-1, 1],$$

and we denote by

$$S^\pm = \{ y \in \mathbb{R}^n : \bar{y} \in Y, y_n = \pm 1 \}$$

the upper and lower boundaries of $Z$. The outer unit normal at the boundaries of the domains $\Omega$ and $\Omega_M^\varepsilon$ is denoted by $\nu$. The restrictions of functions defined on $\Omega$ to the subdomains $\Omega^+_\varepsilon$, $\Omega^-_\varepsilon$, and $\Omega^M_\varepsilon$ are denoted by the superscripts $+$, $-$, and $M$, respectively.

In the domain $\Omega$ we consider the following system of reaction-diffusion equations
for the unknown vector \( u_\varepsilon = (u_{1\varepsilon}, \ldots, u_{m\varepsilon}) : (0, T) \times \Omega \to \mathbb{R}^m \),

\[
\begin{align*}
\partial_t u^+_j - D^+_j \Delta u^+_j &= f_j(x, u^+_\varepsilon) \quad &\text{in } (0, T) \times \Omega^+_\varepsilon, \\
\partial_t u^-_j - D^-_j \Delta u^-_j &= f_j(x, u^-_\varepsilon) \quad &\text{in } (0, T) \times \Omega^-_\varepsilon, \\
\frac{1}{\varepsilon} \partial_t u^M_j - \nabla \cdot \left( \varepsilon D^M_j \left( \frac{x}{\varepsilon} \right) \nabla u^M_j \right) &= \frac{1}{\varepsilon} g_j \left( \frac{x}{\varepsilon}, u^M_\varepsilon \right) \quad &\text{in } (0, T) \times \Omega^M_\varepsilon,
\end{align*}
\]

subjected to the boundary conditions

\[
\begin{align*}
u_j^\pm &= u^\pm_D \quad &\text{on } \partial_D \Omega^\pm, \\
\nabla u_j \cdot \nu &= 0 \quad &\text{on } \partial_N \Omega
\end{align*}
\]

and initial conditions

\[
\begin{align*}
u_j (0, x) &= \begin{cases} 
U_0(x), & x \in \Omega^+_\varepsilon, \\
U^M_0 \left( x, \frac{x}{\varepsilon} \right), & x \in \Omega^M_\varepsilon, \\
U_0(x), & x \in \Omega^-_\varepsilon.
\end{cases}
\end{align*}
\]

Here the boundaries \( \partial_D \Omega^\pm \), respectively, \( \partial_N \Omega \), are defined as follows:

\[
\begin{align*}
\partial_D \Omega^\pm &= \partial \Omega \cap \{ x \in \mathbb{R}^n, x_n = \pm H \}, \\
\partial_N \Omega &= \partial \Omega \setminus \{ \partial_D \Omega^+ \cup \partial_D \Omega^- \}.
\end{align*}
\]

On the interfaces \( S^+_\varepsilon \) and \( S^-_\varepsilon \) we require the natural transmission conditions, i.e., the continuity of the solutions and of the normal fluxes:

\[
\begin{align*}
u_j^\pm &= \nu_j^M \quad &\text{on } S^\pm_\varepsilon, \\
D_j^\pm \nabla u^\pm_j \cdot \nu &= \varepsilon D_j^M \left( \frac{x}{\varepsilon} \right) \nabla u^M_j \cdot \nu \quad &\text{on } S^\pm_\varepsilon.
\end{align*}
\]

**Assumptions on the data.** For the diffusion coefficients \( D_j : \Omega \to \mathbb{R} \), \( j = 1, \ldots, m \), given by

\[
D_j(x) = \begin{cases} 
D^+_j, & x \in \Omega^+_\varepsilon, \\
D^M_j \left( \frac{x}{\varepsilon} \right), & x \in \Omega^M_\varepsilon, \\
D^-_j, & x \in \Omega^-_\varepsilon,
\end{cases}
\]

we assume the following:

- \( D^+_j > 0, D^-_j > 0, j = 1, \ldots, m \).
- \( D^M_j \) is defined on the standard cell \( \mathbb{Z} \) and belongs to \( C^1_{\text{per}}([0,1]^{n-1}, C^1([-1,1])) \).
- We also assume that it is strictly positive.

Concerning the reaction terms we suppose the following:

- \( f = f(x, z) : \Omega \times \mathbb{R}^m \to \mathbb{R}^m \) is continuous and Lipschitz continuous with respect to \( z \), with a Lipschitz constant independent of \( x \).
- \( g = g(y, z) : [0,1]^{n-1} \times [-1,1] \times \mathbb{R}^m \to \mathbb{R}^m \) is continuous, Lipschitz continuous in \( z \), and periodic in \( \bar{y} = (y_1, \ldots, y_{n-1}) \).

The assumptions on the reaction terms imply that there exist positive constants \( c_1 \) and \( c_2 \) such that for \( j = 1, \ldots, m \)

\[
\begin{align*}
|f_j(x, z)| &\leq c_1 (1 + |z|) \quad \text{for all } z \in \mathbb{R}^m, x \in \Omega, \\
|g_j(y, z)| &\leq c_2 (1 + |z|) \quad \text{for all } z \in \mathbb{R}^m, y \in Z.
\end{align*}
\]
Additionally, we have to impose on \( f \) and \( g \) structural conditions which guarantee \( L^\infty \)-estimates of the solutions \( u_\varepsilon \). A possible choice of such conditions is given in the following.

Let \( M_j \in \mathbb{R}, M_j > 0, j = 1, \ldots, m \), be given. We consider

\[
\begin{align*}
(2.7) & \quad f_j(\cdot, z) \leq A_j z_j \quad \text{for } z_j \geq M_j, \\
(2.8) & \quad g_j(\cdot, z) \leq A_j z_j \quad \text{for } z_j \geq M_j,
\end{align*}
\]

where \( A_j \in \mathbb{R}, A_j \geq 0, j = 1, \ldots, m \). We also require that

\[
\begin{align*}
(2.9) & \quad \sum_{j=1}^{m} f_j(\cdot, z)(z_j^-) \leq C \sum_{j=1}^{m} |z_j^-|^2, \\
(2.10) & \quad \sum_{j=1}^{m} g_j(\cdot, z)(z_j^-) \leq C \sum_{j=1}^{m} |z_j^-|^2,
\end{align*}
\]

where \( (z_j^-) = \min\{z_j, 0\} \). For the initial functions we assume that \( U_0 \in H^2(\Omega^+, \mathbb{R}^m) \cap H^1(\Omega^-, \mathbb{R}^m) \), \( U_0^M \in H^2(\Sigma \times ]-1, 1[ \times \mathbb{R}^m) \), such that

\[
\frac{1}{\sqrt{\varepsilon}} \left\| U_0^M \left( \cdot, \frac{x}{\varepsilon} \right) \right\|_{L^2(\Omega^+, \mathbb{R}^m)} + \sqrt{\varepsilon} \left\| \nabla U_0^M \left( \cdot, \frac{x}{\varepsilon} \right) \right\|_{L^2(\Omega^+, \mathbb{R}^m)} \leq C,
\]

and that they satisfy the compatibility conditions

\[
\begin{align*}
U_0(x) & = U_0^M \left( \bar{x}, \frac{x_n}{\varepsilon} \right) \quad \text{on } S^\pm_\varepsilon, \\
D_j^\pm \nabla U_{j0} \cdot \nu & = \varepsilon D_j^M \left( \frac{x}{\varepsilon} \right) \nabla U_{j0}^M \cdot \nu \quad \text{on } S^\pm_\varepsilon, \\
U_0(x) & = u_D(0, x) \quad \text{on } \partial D \Omega^\pm.
\end{align*}
\]

For the Dirichlet boundary data we require

\[
\begin{align*}
(2.13) & \quad u_D \in L^2((0, T), H^2(\Omega, \mathbb{R}^m)), \ \text{supp}(u_D) \cap \Omega^M_\varepsilon = \emptyset, \\
(2.14) & \quad \partial_t u_D^M \in L^2((0, T), H^1(\Omega^M_\varepsilon)) \cap L^\infty((0, T) \times \Omega, \mathbb{R}^m).
\end{align*}
\]

In order to obtain the \( L^\infty \)-estimates for the solution \( u_\varepsilon \), we have to assume that the initial and boundary functions also satisfy corresponding bounds. For the example of reaction terms given above, we assume that

\[
\begin{align*}
(2.15) & \quad 0 \leq U_{j0} \leq M_j, \ 0 \leq U_{j0}^M \leq M_j, \ 0 \leq u_{jD} \leq M_j, \quad j = 1, \ldots, m.
\end{align*}
\]

**Variational formulation of the microscopic problem.** We denote by \( X \) the function space

\[
X = \{ u \in H^1(\Omega, \mathbb{R}^m) : u = 0 \text{ on } \partial D \Omega^+ \cup \partial D \Omega^- \}.
\]

The variational formulation of problem (2.1)–(2.4) is given as follows: Find \( u_\varepsilon : (0, T) \times \Omega \rightarrow \mathbb{R}^m \), such that \( u_\varepsilon - u_D \in L^2((0, T), \Omega), \ \partial_t (u_\varepsilon - u_D) \in L^2((0, T), L^2(\Omega)), \) and for all \( \varphi \in L^2((0, T), X) \) and a.e. \( t \in (0, T) \) we have

\[
\begin{align*}
(2.16) & \quad \int_{\Omega^+_t} \partial_t u^M_{j\varepsilon} \varphi_j \, dx + \int_{\Omega^-_t} \partial_t u^-_{j\varepsilon} \varphi_j \, dx + \frac{1}{\varepsilon} \int_{\Omega^+_M} \partial_t u^M_{j\varepsilon} \varphi_j \, dx \\
& \quad + D_j^+ \int_{\Omega^+_t} \nabla u^+_j \nabla \varphi_j \, dx + D_j^- \int_{\Omega^-_t} \nabla u^-_j \nabla \varphi_j \, dx + \int_{\Omega^+_M} \varepsilon D_j^M \nabla u^M_j \nabla \varphi_j \, dx \\
& \quad = \int_{\Omega^+_t} f_j(x, u^+_\varepsilon) \varphi_j \, dx + \int_{\Omega^-_t} f_j(x, u^-_{\varepsilon}) \varphi_j \, dx + \frac{1}{\varepsilon} \int_{\Omega^+_M} g_j \left( \frac{x}{\varepsilon}, u^M_{\varepsilon} \right) \varphi_j \, dx
\end{align*}
\]
Fig. 2.3. The structure of the domain $\Omega$ in the limit $\varepsilon = 0$.  

\begin{align*}
\text{and} \\
u_\varepsilon(0,x) = \begin{cases} 
U_0(x), \quad x \in \Omega^+_\varepsilon, \\
U_0^M(\bar{x}, \frac{x_n}{\varepsilon}), \quad x \in \Omega^M_\varepsilon, \\
U_0(x), \quad x \in \Omega^-_\varepsilon.
\end{cases}
\end{align*}

The existence and uniqueness of weak solutions for the problem (2.1)–(2.4) for every fixed $\varepsilon > 0$ is standard, e.g., by using the Galerkin method based on estimates similar to those in section 3.

Our aim is now to study the behavior of the solutions $u_\varepsilon$ for small values of the parameter $\varepsilon$. We will do this by studying the asymptotic behavior of the sequence $u_\varepsilon$ for $\varepsilon \to 0$.

When $\varepsilon$ tends to zero, the thin layer $\Omega^M_\varepsilon$ approaches the interface $\Sigma$. The domains $\Omega^+_\varepsilon$ and $\Omega^-_\varepsilon$ tend to the domains $\Omega^+$ and $\Omega^-$, respectively, defined below:

\begin{align*}
\Omega^+ &= [0,1]^{n-1} \times [0,H], \\
\Omega^- &= [0,1]^{n-1} \times [-H,0].
\end{align*}

Thus the macroscopic limit of the sequence $u_\varepsilon$ (if it exists) will be defined on the domain $\Omega$ consisting of (see Figure 2.3)

$$
\Omega = \Omega^+ \cup \Omega^- \cup \Sigma.
$$

2.2. Main results. From the a priori estimates (given in section 3), it is obvious that different convergence concepts have to be used for studying the asymptotic behavior of the solutions $u_\varepsilon$ in the bulk and thin layer regions. Whereas in $\Omega^\pm_\varepsilon$ classical compactness results can be used, in $\Omega^M_\varepsilon$ compactness needs to be considered with respect to the weak and strong two-scale convergences adapted to the thin layer. The concepts of multiscale convergence in the weak and strong sense, also for thin and
periodic structures, are crucial for formulating and proving the main results of this paper. For the definition and properties of two-scale convergence for thin heterogeneous layers, see section 4.

In the following two propositions, we state the convergence results in the bulk and thin layer regions as well as the convergence of the traces on the interfaces $S^+$.

For the layer region, we obtain in a first step weak two-scale convergence.

**Proposition 2.1.** There exists a subsequence denoted again $u_\varepsilon$ and limit functions $u_j^\pm_0 \in \mathcal{L}^2((0,T),H^1(\Omega^\pm,\mathbb{R}^m))$, with $\partial_t u_j^\pm_0 \in \mathcal{L}^2((0,T),\mathcal{L}^2(\Omega^\pm,\mathbb{R}^m))$, and $u_\varepsilon^M \in \mathcal{L}^2((0,T) \times \Sigma, H^1_{\text{per}}(Y,H^1([-1,1]))^m$, with $\partial_t u_\varepsilon^M \in \mathcal{L}^2((0,T) \times \Sigma, \mathcal{L}^2(Z))^m$, such that

1. $\chi_{\Omega^\pm_\varepsilon} u_\varepsilon \rightharpoonup u_j^\pm_0$ strongly in $\mathcal{L}^2((0,T),\mathcal{L}^2(\Omega^\pm))$;
2. $\chi_{\Omega^\pm_\varepsilon} \nabla u_\varepsilon \rightharpoonup \nabla u_j^\pm_0$ weakly in $\mathcal{L}^2((0,T),\mathcal{L}^2(\Omega^\pm))$;
3. $\chi_{\Omega^\pm_\varepsilon} \partial_t u_\varepsilon \rightharpoonup \partial_t u_j^\pm_0$ weakly in $\mathcal{L}^2((0,T),\mathcal{L}^2(\Omega^\pm))$;
4. $u_j^\pm_\varepsilon \rightharpoonup u_j^\pm_0(t,\bar{x},\bar{y})$ weakly in the two-scale sense;
5. $\varepsilon \nabla u_j^M \rightharpoonup \nabla_p u_j^M(t,\bar{x},\bar{y})$ weakly in the two-scale sense;
6. $\partial_t u_j^M \rightharpoonup \partial_t u_j^M(t,\bar{x},\bar{y})$ weakly in the two-scale sense.

Furthermore, we have

\begin{equation}
(2.19) \quad u_j^M(t,\bar{x},\bar{y},\pm 1) = u_0^\pm(t,\bar{x},0) \quad \text{a.e.} \quad (t,\bar{x}) \in (0,T) \times \Sigma, \bar{y} \in Y.
\end{equation}

**Proposition 2.2.** There exists a subsequence denoted again by $u^\pm_\varepsilon$, such that

\begin{equation}
(2.20) \quad \tilde{u}^\pm_\varepsilon \rightharpoonup u^\pm_0 \text{ strongly in } \mathcal{L}^2((0,T) \times \Sigma),
\end{equation}

\begin{equation}
(2.21) \quad \lim_{\varepsilon \to 0} \int_0^T \int_{\Sigma^\pm_\varepsilon} u_j^\pm_\varepsilon(t,x)\varphi_j(t,\bar{x},\bar{y}) \, dx \, dt = \int_0^T \int_{\Sigma} u^\pm_0(t,\bar{x},0)\varphi_j(t,\bar{x},\bar{y},\pm 1) \, d\bar{y} \, d\bar{x} \, dt
\end{equation}

for all $\varphi_j \in C^\infty([0,T] \times \Sigma, C^\infty_{\text{per}}(Y,C^\infty([-1,1])))$, where $u_j^\pm_0$ are the limit functions given in Proposition 2.1, and the scaled functions $\tilde{u}^\pm_\varepsilon$ are defined in (5.2).

A central contribution of this paper is formulated in Theorem 2.3 below, where a strong two-scale convergence of the solutions $u_\varepsilon$ in the layer $\Omega^M_\varepsilon$ is obtained. This strong two-scale convergence is not based on extension properties but uses the reaction-diffusion equations in the layer and the compactness of the traces on $S^+$.

**Theorem 2.3.** The extension $\tilde{u}^M_\varepsilon$ of $u^M_\varepsilon$ to $\tilde{\Omega}_\varepsilon^M$ defined in section 5.4 can be estimated in terms of its boundary values on $\tilde{S}^\pm_\varepsilon$ and its initial values $\tilde{U}_0^M$ in the following way: Fix $h \in (0,\frac{1}{4})$ and assume $l \in \mathbb{Z}^{n-1}$ such that $|l| < h$. Then there exists a constant $C$ independent of $\varepsilon$ and $l$, such that

\begin{equation}
(2.22) \quad \frac{1}{\sqrt{\varepsilon}}||u^M_\varepsilon(t,x+(l,0)\varepsilon) - \tilde{u}^M_\varepsilon(t,x)||_{\mathcal{L}^2((0,T) \times \Omega^M_\varepsilon,\mathbb{R}^m)}
\end{equation}

\begin{equation}
\leq C \left(||\tilde{u}^M_\varepsilon(t,x+(l,0)\varepsilon) - \tilde{u}^+_\varepsilon(t,x)||_{\mathcal{L}^2((0,T) \times \tilde{S}^+_\varepsilon,\mathbb{R}^m)} + ||u^M_\varepsilon(t,x+(l,0)\varepsilon) - \tilde{u}^-_\varepsilon(t,x)||_{\mathcal{L}^2((0,T) \times \tilde{S}^-_\varepsilon,\mathbb{R}^m)}
\right)
\end{equation}

\begin{equation}
\leq \frac{1}{\sqrt{h}} \left(||\tilde{U}_0^M(\bar{x} + l\varepsilon, \frac{x_n}{\varepsilon}) - \tilde{U}_0^M(\bar{x}, \frac{x_n}{\varepsilon})||_{\mathcal{L}^2(\tilde{\Omega}_0^M,\mathbb{R}^m)} + C \frac{\varepsilon}{h} + Ch^\varepsilon.\right)
\end{equation}
Thus, up to subsequence, \( u^M_n \) converges also strongly in the two-scale sense to the limit function \( u^M_0 \).

The limit functions satisfy the macroscopic problem formulated in the following theorem.

**Theorem 2.4.** The limit functions \( u^\pm_0 \) given in Proposition 2.1 satisfy in a distributional sense the initial boundary value problem on \( \Omega^+ \) and \( \Omega^- \), respectively,

\[
\begin{align*}
(2.23) \quad & \partial_t u^+_j + D^+_j \Delta u^+_j = f_j(x, u^+_0), \quad (t, x) \in (0, T) \times \Omega^+, \\
(2.24) \quad & \partial_t u^-_j - D^-_j \Delta u^-_j = f_j(x, u^-_0), \quad (t, x) \in (0, T) \times \Omega^-, \\
(2.25) \quad & u^+_0(t, x) = u^-_0, \quad (t, x) \in (0, T) \times \partial_D \Omega^\pm, \\
(2.26) \quad & \frac{\partial u^\pm_0}{\partial \nu} = 0, \quad (t, x) \in (0, T) \times \partial_N \Omega^\pm, \\
(2.27) \quad & u^\pm_0(0, x) = U_0(x), \quad x \in \Omega^\pm,
\end{align*}
\]

Together with the effective transmission conditions on the interface \( \Sigma \),

\[
(2.28) \quad [u_0]\Sigma(t, \bar{x}) = \int_Z (g_j(y, u^M_0(t, \bar{x}, y)) - \partial_t u^M_0(t, \bar{x}, y)) \eta_j(y) \, dy, \\
\quad + D^+_j \eta^+_j \partial_n u^+_0(t, \bar{x}, 0) - D^-_j \eta^-_j \partial_n u^-_0(t, \bar{x}, 0), \\
\quad + \int_Z (\partial_t u^M_0(t, \bar{x}, y) - g_j(y, u^M_0(t, \bar{x}, y))) \, dy.
\]

The limit function \( u^M_0 \), which enters the transmission conditions, is the weak solution of the local problem

\[
\begin{align*}
(2.30) \quad & \partial_t u^M_0(t, \bar{x}, y) - \nabla_y \cdot (D^M_j(y) \nabla_y u^M_0(t, \bar{x}, y)) \\
& \quad = g_j(y, u^M_0(t, \bar{x}, y)) \text{ in } (0, T) \times Z, \\
(2.31) \quad & u^M_0(t, \bar{x}, y) = u^\pm_0(t, \bar{x}, 0) \text{ on } (0, T) \times S^\pm, \\
(2.32) \quad & u^M_0 \text{ is periodic in } Y, \\
(2.33) \quad & u^M_0(0, \bar{x}, y) = U^M_0(\bar{x}, y_n) \text{ in } Z
\end{align*}
\]

for a.e. \( \bar{x} \in \Sigma. \) The constants \( \eta^\pm = (\eta^+_1, \ldots, \eta^+_m) \) are the constant boundary values on \( S^\pm \) of the solution \( \eta = (\eta_1, \ldots, \eta_m) \) to the following boundary value problem on the standard cell: Find

\[
\eta = (\eta_1, \ldots, \eta_m) \in V := \{ \varphi \in H^1(Z, \mathbb{R}^m), \varphi \text{ periodic in } Y, \varphi \equiv \text{ const on } S^+ \cup S^- \}
\]

such that \( \frac{1}{|Z|} \int_Z \eta(y) \, dy = 0 \) and, for all \( \varphi \in V, \)

\[
(2.34) \quad \int_Z D^M_j(y) \nabla \eta_j(y) \nabla \varphi_j(y) \, dy = \int_{S^+} \varphi_j(y) \, ds - \int_{S^-} \varphi_j(y) \, ds.
\]

In the final step, we prove uniqueness for the macroscopic problem, and therefore we obtain that the sequence of solutions to the microscopic problems converges to the solution of the macroscopic problem in the corresponding topology on every subdomain.

**Theorem 2.5.** The solution \( (u^+_0, u^-_0, u^M_0) \) of the macroscopic system (2.23)–(2.33) is unique.
3. A priori estimates for the microscopic model. In order to get some information about the compactness properties of the sequence $u_\varepsilon$, we have to control the dependence of the solutions on the scale parameter $\varepsilon$.

**Lemma 3.1.** For the solutions of problem (2.1)–(2.4), the following estimates hold, with a generic constant $C$ independent of $\varepsilon$:

\begin{equation}
|u_{j\varepsilon}^\pm|_{L^\infty((0,T),H^1(\Omega_\varepsilon^\pm))} \leq C, \tag{3.1}
\end{equation}

\begin{equation}
\frac{1}{\sqrt{\varepsilon}}|u_{j\varepsilon}^M|_{L^\infty((0,T),L^2(\Omega_j^M))} + \varepsilon|\nabla u_{j\varepsilon}^M|_{L^\infty((0,T),L^2(\Omega_j^M))} \leq C, \tag{3.2}
\end{equation}

\begin{equation}
|\partial_t u_{j\varepsilon}^\pm|_{L^2((0,T),L^2(\Omega_\varepsilon^\pm))} \leq C, \quad \frac{1}{\sqrt{\varepsilon}}|\partial_t u_{j\varepsilon}^M|_{L^2((0,T),L^2(\Omega_j^M))} \leq C. \tag{3.3}
\end{equation}

In the case that the reaction terms satisfy the additional structural conditions (2.7)–(2.9), we obtain the following pointwise bounds for the solution:

\begin{equation}
0 \leq u_{j\varepsilon} \leq M e^{A_j t} \quad \text{for a.e. } (t,x) \in (0,T) \times \Omega. \tag{3.4}
\end{equation}

**Proof.** Let us first set

$$U_\varepsilon = u_\varepsilon - u_D = \begin{cases} u_{j\varepsilon}^\pm - u_D, & x \in \Omega_\varepsilon^\pm, \\ u_j^M, & x \in \Omega_j^M, \end{cases}$$

and use it as a test function in (2.16). We obtain

\begin{align*}
&\int_{\Omega_\varepsilon^\pm} \partial_t U_{j\varepsilon}^\pm(t)U_{j\varepsilon}^\pm(t) \, dx + \int_{\Omega_\varepsilon^\pm} \partial_t u_{j,D}(t)U_{j\varepsilon}^\pm(t) \, dx \\ &+ D_j \int_{\Omega_\varepsilon^\pm} \nabla U_{j\varepsilon}^\pm(t)\nabla U_{j\varepsilon}^\pm(t) \, dx + D_j \int_{\Omega_\varepsilon^\pm} \nabla u_{j,D}(t)\nabla U_{j\varepsilon}^\pm(t) \, dx \\ &+ \frac{1}{\varepsilon} \int_{\Omega_j^M} \partial_t u_j^M(t)u_j^M(t) \, dx + \frac{1}{\varepsilon} \int_{\Omega_j^M} D_j \left( \frac{x}{\varepsilon} \right) \nabla u_j^M(t)\nabla u_j^M(t) \, dx \\ &+ \int_{\Omega_j^M} f_j(x,u_j^\pm(t,x))U_{j\varepsilon}^\pm(t) \, dx + \frac{1}{\varepsilon} \int_{\Omega_j^M} g_j \left( \frac{x}{\varepsilon},u_j^M(t,x) \right) u_j^M(t) \, dx
\end{align*}

\begin{equation}
\leq C_1 \int_{\Omega_\varepsilon^\pm} (1 + |u_{j\varepsilon}^\pm|) |U_{j\varepsilon}^\pm(t)| \, dx + \frac{C_2}{\varepsilon} \int_{\Omega_j^M} (1 + |u_j^M|) |u_j^M(t)| \, dx
\end{equation}

a.e. in $(0,T)$. Here the integrals over $\Omega_\varepsilon^\pm$ stand for the sum of the integrals over $\Omega_\varepsilon^+$ and $\Omega_\varepsilon^-$. For the last inequality we used the growth conditions (2.5) and (2.6) on the reaction terms. Adding up the estimates in (3.5) for $j = 1, \ldots, m$ and integrating with respect to time yields

\begin{align*}
&\frac{1}{2}||U_\varepsilon^\pm(t)||_{L^2(\Omega_\varepsilon^\pm,R^m)}^2 + \frac{1}{2\varepsilon}||u_j^M(t)||_{L^2(\Omega_j^M,R^m)}^2 \\
&+ \int_0^t \int_{\Omega_\varepsilon^\pm} |\nabla U_\varepsilon^\pm|^2 \, dx \, dt + \varepsilon \int_0^t \int_{\Omega_j^M} |\nabla u_j^M(t)|^2 \, dx \, dt \\
&\leq C \left( \int_0^t \int_{\Omega_\varepsilon^\pm} |\partial_t u_D(t)U_\varepsilon^\pm| \, dx \, dt + \int_0^t \int_{\Omega_\varepsilon^\pm} |\nabla u_D|\nabla U_\varepsilon^\pm| \, dx \, dt \\
&+ \int_0^t \int_{\Omega_\varepsilon^\pm} |U_\varepsilon^\pm|^2 \, dx \, dt \right) \\
&+ C \left( 1 + \int_0^t \int_{\Omega_j^M} |U_\varepsilon^\pm|^2 \, dx \, dt + \frac{1}{\varepsilon} \int_0^t \int_{\Omega_j^M} |u_j^M|^2 \, dx \, dt \right) \\
&+ \frac{1}{2}||U_\varepsilon^\pm(0)||_{L^2(\Omega_\varepsilon^\pm,R^m)}^2 + \frac{1}{2\varepsilon}||u_j^M(0)||_{L^2(\Omega_j^M,R^m)}^2.
\end{align*}
Here we also used the regularity properties (2.14) of the boundary data \(u_D\). To estimate the second term on the right-hand side, we make use of the inequality

\[
2ab \leq \delta a^2 + \frac{1}{\delta} b^2
\]

to get

\[
\int_0^t \int_{\Omega^\pm} |\nabla u_D \nabla U^\pm| \, dx \, dt \leq C \left( \frac{1}{\delta} + \delta \|\nabla U^\pm\|^2_{L^2((0,T) \times \Omega^\pm, \mathbb{R}^m)} \right).
\]

If \(\delta\) is small enough, the term involving \(\nabla U^\pm\) can be absorbed on the left-hand side. Now, using Gronwall’s lemma and the assumptions (2.11) on the initial conditions, we obtain

\[
\|U^\pm\|_{L^\infty((0,T), L^2(\Omega^\pm, \mathbb{R}^m))} + \frac{1}{\sqrt{\varepsilon}} \|u^M\|_{L^\infty((0,T), L^2(\Omega^M, \mathbb{R}^m))} \leq C
\]

and

\[
\|\nabla U^\pm(t)\|_{L^2((0,T), L^2(\Omega^\pm, \mathbb{R}^m))} + \sqrt{\varepsilon} \|u^M(t)\|_{L^2((0,T), L^2(\Omega^M, \mathbb{R}^m))} \leq C.
\]

To obtain the \(L^\infty\)-estimates with respect to time for the gradients and the estimates for the time derivatives, we take \(\varphi = \partial_t U_\varepsilon\) as a test function in (2.16). It yields

\[
\int_{\Omega^M} \partial_t U^\pm \frac{x}{\varepsilon} \partial_t U^\pm \frac{x}{\varepsilon} \, dx + \int_{\Omega^M} \partial_t u^M \frac{x}{\varepsilon} \partial_t u^M \frac{x}{\varepsilon} \, dx
\]

\[
+ D_j^\pm \int_{\Omega^M} \nabla U^\pm \frac{x}{\varepsilon} \partial_t U^\pm \frac{x}{\varepsilon} \, dx + D_j^\pm \int_{\Omega^M} \nabla u^M \frac{x}{\varepsilon} \partial_t U^\pm \frac{x}{\varepsilon} \, dx
\]

\[
+ \frac{1}{\varepsilon} \int_{\Omega^M} \partial_t u^M \frac{x}{\varepsilon} \partial_t u^M \frac{x}{\varepsilon} \, dx + \frac{1}{\varepsilon} \int_{\Omega^M} \nabla u^M \frac{x}{\varepsilon} \partial_t u^M \frac{x}{\varepsilon} \, dx
\]

\[
= \int_{\Omega^M} f_j(x, u^\varepsilon) \partial_t U^\pm \frac{x}{\varepsilon} \, dx + \frac{1}{\varepsilon} \int_{\Omega^M} g_j(x, u^\varepsilon) \partial_t u^M \frac{x}{\varepsilon} \, dx
\]

a.e. on \((0,T)\). First, we have to transform the energy integral on \(\Omega^M\) as follows:

\[
\varepsilon \int_{\Omega^M} D_j^M \left( \frac{x}{\varepsilon} \right) \nabla u^M \frac{x}{\varepsilon} \partial_t u^M \frac{x}{\varepsilon} \, dx = \frac{d}{dt} \int_{\Omega^M} \varepsilon D_j^M \left( \frac{x}{\varepsilon} \right) \nabla u^M \frac{x}{\varepsilon} \partial_t u^M \frac{x}{\varepsilon} \, dx
\]

\[
- \varepsilon \int_{\Omega^M} D_j^M \left( \frac{x}{\varepsilon} \right) \nabla \partial_t u^M \frac{x}{\varepsilon} \partial_t u^M \frac{x}{\varepsilon} \, dx.
\]

Thus,

\[
(3.7)
\]

\[
\varepsilon \int_{\Omega^M} D_j^M \left( \frac{x}{\varepsilon} \right) \nabla u^M \frac{x}{\varepsilon} \partial_t u^M \frac{x}{\varepsilon} \, dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega^M} \varepsilon D_j^M \left( \frac{x}{\varepsilon} \right) \nabla u^M \frac{x}{\varepsilon} \partial_t u^M \frac{x}{\varepsilon} \, dx.
\]

Adding up the equations for \(j = 1, \ldots, m\), taking into account (3.7) and the growth conditions (2.5), (2.6), and integrating with respect to time, we obtain

\[
(3.8)
\]

\[
\|\partial_t U^\pm\|^2_{L^2((0,T), L^2(\Omega^\pm, \mathbb{R}^m))} + \frac{1}{\varepsilon} \|\partial_t u^M\|^2_{L^2((0,T), L^2(\Omega^M, \mathbb{R}^m))}
\]

\[
+ \|\nabla U^\pm(t)\|^2_{L^2(\Omega^\pm, \mathbb{R}^m)} + \varepsilon \|\nabla u^M(t)\|^2_{L^2(\Omega^M, \mathbb{R}^m)}
\]

\[
\leq C \left( 1 + \|U^\pm\|^2_{L^2((0,T), L^2(\Omega^\pm, \mathbb{R}^m))} + \frac{1}{\varepsilon} \|u^M\|^2_{L^2((0,T), L^2(\Omega^M, \mathbb{R}^m))} \right)
\]

\[
+ C \|\nabla U^\pm\|^2_{L^2((0,T), L^2(\Omega^\pm, \mathbb{R}^m))}.
\]
Using the estimates obtained in the first part of the proof, it follows that the right-hand side of (3.8) is bounded independently of $\varepsilon$. Thus estimates (3.1)–(3.3) are proved.

Now, it remains to show the $L^\infty$-bounds for the solution under the hypothesis (2.7)–(2.10) on the reaction terms. We first show positivity of the solutions. Let us test our system (2.16) with the test function $\varphi$ given by

$$
\varphi_j = (u_{je})_+ = \min\{u_{je}, 0\} \quad \text{a.e. on } [0, T] \times \Omega.
$$

Due to the the assumptions (2.15), our test function has zero boundary values on the parabolic boundary. Thus we obtain

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega^+_T} |(u_{je})_+|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega^+_T} |(u_{je})_-|^2 \, dx + \frac{1}{2\varepsilon} \frac{d}{dt} \int_{\Omega^M} |(u_{je}^M)_+|^2 \, dx
$$

$$
+ D_j^+ \int_{\Omega^+_T} \nabla (u_{je})_+ \cdot \nabla (u_{je})_- \, dx + D_j^- \int_{\Omega^+_T} \nabla (u_{je})_- \cdot \nabla (u_{je})_- \, dx + \int_{\Omega^M} \varepsilon D_j^M \nabla (u_{je}^M)_- \cdot \nabla (u_{je}^M)_- \, dx
$$

$$
= \int_{\Omega^+_T} f_j(x, u_{je}^+)(u_{je})_- + \int_{\Omega^+_T} f_j(x, u_{je}^-)(u_{je})_- + \frac{1}{\varepsilon} \int_{\Omega^M} g_j \left( \frac{2}{\varepsilon}, u_{je}^M \right) (u_{je}^M)_-.
$$

Integrating with respect to time, adding up the equations for $j = 1, \ldots, m$, and using the assumptions (2.9), (2.10) on the reaction terms leads to

$$
\int_{\Omega^+_T} \sum_{j=1}^m |(u_{je})_+(t)|^2 \, dx + \int_{\Omega^+_T} \sum_{j=1}^m |(u_{je})_-(t)|^2 \, dx + \frac{1}{\varepsilon} \int_{\Omega^M} \sum_{j=1}^m |(u_{je}^M)_-(t)|^2 \, dx
$$

$$
\leq C \int_0^T \left\{ \int_{\Omega^+_T} \sum_{j=1}^m |(u_{je})_+|^2 \, dx + \int_{\Omega^+_T} \sum_{j=1}^m |(u_{je})_-|^2 \, dx + \frac{1}{\varepsilon} \int_{\Omega^M} \sum_{j=1}^m |(u_{je}^M)_-|^2 \, dx \right\} dt.
$$

Now, Gronwall's inequality implies that $(u_{je})_- = 0$. Thus the positivity of the solution is proved. To obtain the upper bound, we first test (2.16) with the test function

$$
\varphi_j(t, x) = e^{-\lambda t} \psi_j(t, x),
$$

where $\psi \in L^2((0, T), H^1(\Omega))$ and has zero boundary values on the parabolic boundary. We obtain

$$
\int_{\Omega^+_T} \partial_t u_{je}^+ e^{-\lambda t} \psi_j \, dx + \int_{\Omega^+_T} \partial_t u_{je}^- e^{-\lambda t} \psi_j \, dx + \frac{1}{\varepsilon} \int_{\Omega^M} \partial_t u_{je}^M e^{-\lambda t} \psi_j \, dx
$$

$$
+ D_j^+ \int_{\Omega^+_T} e^{-\lambda t} \nabla u_{je}^+ \cdot \nabla \psi_j \, dx + D_j^- \int_{\Omega^+_T} e^{-\lambda t} \nabla u_{je}^- \cdot \nabla \psi_j \, dx
$$

$$
+ \int_{\Omega^M} \varepsilon D_j^M e^{-\lambda t} \nabla u_{je}^M \cdot \nabla \psi_j \, dx = \int_{\Omega^+_T} f_j(x, u_{je}^+ e^{-\lambda t} \psi_j \, dx
$$

$$
+ \int_{\Omega^+_T} f_j(x, u_{je}^-) e^{-\lambda t} \psi_j \, dx + \frac{1}{\varepsilon} \int_{\Omega^M} g_j \left( \frac{2}{\varepsilon}, u_{je}^M \right) e^{-\lambda t} \psi_j \, dx.
$$

Now we intend to set

$$
\psi_j = (e^{-\lambda t} u_{je} - M_j)_+ = \max\{e^{-\lambda t} u_{je} - M_j, 0\} \quad \text{a.e. on } [0, T] \times \Omega.
$$

Therefore, we write the terms containing the time derivative as

$$
\int_{\Omega^+_T} \partial_t u_{je}^+ e^{-\lambda t} \psi_j \, dx = \int_{\Omega^+_T} \partial_t (e^{-\lambda t} u_{je}^+ - M_j) \psi_j \, dx + \int_{\Omega^+_T} A_j e^{-\lambda t} u_{je}^+ \psi_j \, dx,
$$

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and analogously the terms on $\Omega^-_\varepsilon$ and $\Omega^M_\varepsilon$. We obtain
\begin{equation}
(3.9) \quad \frac{1}{2} \frac{d}{dt} \int_{\Omega^+_\varepsilon} |(e^{-A_j^t} u^+_{j\varepsilon} - M_j)_+|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega^-_\varepsilon} |(e^{-A_j^t} u^-_{j\varepsilon} - M_j)_+|^2 \, dx \\
+ \frac{1}{2\varepsilon} \frac{d}{dt} \int_{\Omega^M_\varepsilon} |(e^{-A_j^t} u^M_{j\varepsilon} - M_j)_+|^2 \, dx + \int_{\Omega^+_\varepsilon} A_j e^{-A_j^t} u^+_{j\varepsilon} (e^{-A_j^t} u^+_{j\varepsilon} - M_j)_+ \, dx \\
+ \int_{\Omega^-_\varepsilon} A_j e^{-A_j^t} u^-_{j\varepsilon} (e^{-A_j^t} u^-_{j\varepsilon} - M_j)_+ \, dx + \frac{1}{\varepsilon} \int_{\Omega^M_\varepsilon} A_j e^{-A_j^t} u^M_{j\varepsilon} (e^{-A_j^t} u^M_{j\varepsilon} - M_j)_+ \, dx \\
\leq \int_{\Omega^+_\varepsilon} f_j(x, u^+_{\varepsilon}) e^{-A_j^t} (e^{-A_j^t} u^+_{j\varepsilon} - M_j)_+ \, dx + \int_{\Omega^-_\varepsilon} f_j(x, u^-_{\varepsilon}) e^{-A_j^t} (e^{-A_j^t} u^-_{j\varepsilon} - M_j)_+ \, dx \\
+ \frac{1}{\varepsilon} \int_{\Omega^M_\varepsilon} g_j \left( \frac{x}{\varepsilon}, u^M_{\varepsilon} \right) e^{-A_j^t} (e^{-A_j^t} u^M_{j\varepsilon} - M_j)_+ \, dx.
\end{equation}

Now, due to assumptions (2.7), (2.8) on the reaction terms, the right-hand side in the above inequality can be estimated from above by
\begin{align*}
&\int_{\Omega^+_\varepsilon} A_j u^+_{j\varepsilon} e^{-A_j^t} (e^{-A_j^t} u^+_{j\varepsilon} - M_j)_+ \, dx + \int_{\Omega^-_\varepsilon} A_j u^-_{j\varepsilon} e^{-A_j^t} (e^{-A_j^t} u^-_{j\varepsilon} - M_j)_+ \, dx \\
&+ \frac{1}{\varepsilon} \int_{\Omega^M_\varepsilon} A_j u^M_{j\varepsilon} e^{-A_j^t} (e^{-A_j^t} u^M_{j\varepsilon} - M_j)_+ \, dx.
\end{align*}

However, these terms cancel with the corresponding terms on the left-hand side in (3.9), and thus, after integration with respect to time, we get
\begin{align*}
&\int_{\Omega^+_\varepsilon} |(e^{-A_j^t} u^+_{j\varepsilon} - M_j)_+|^2 \, dx + \int_{\Omega^-_\varepsilon} |(e^{-A_j^t} u^-_{j\varepsilon} - M_j)_+|^2 \, dx \\
&+ \frac{1}{\varepsilon} \int_{\Omega^M_\varepsilon} |(e^{-A_j^t} u^M_{j\varepsilon} - M_j)_+|^2 \, dx \leq 0.
\end{align*}

Finally, we have
\[ e^{-A_j^t} u_{j\varepsilon} - M_j \leq 0 \quad \text{a.e. on } [0, T] \times \Omega. \]

This completes the proof. \[ \square \]

4. Two-scale convergence for thin heterogeneous layers. From the a priori estimates in Lemma 3.1, we see that on the subdomain $\Omega^M_\varepsilon$ we cannot use the classic compactness results for passing to the limit when $\varepsilon \to 0$. Here, we have to consider special convergence concepts which are adapted to sequences of functions varying on different scales.

The so-called two-scale convergence was introduced in [1] and [12] in order to handle two-scale phenomena with periodic structure in all space dimensions. Then it was extended to multiple scales [2]; periodic surfaces [13] and measures [17], [4]; thin domains [11]; and stochastic media [6]. For our problem, we need to generalize the concept of two-scale convergence to thin heterogeneous domains. Thus let $G \subset \mathbb{R}^{n-1}$ be a bounded domain and let $Y = [0, 1]^{n-1}$ be the closed unit cube in $\mathbb{R}^{n-1}$. Let $G_\varepsilon$ be a thin domain defined by
\[ G_\varepsilon = G \times ]-\varepsilon, \varepsilon[. \]
Let $\Sigma$ be the interface

$$\Sigma = G \times \{0\}$$

and, as before, let us denote by $Z$ the standard cell

$$Z = Y \times [-1,1].$$

Let $C_{\text{per}}(Y)$ be the space of continuous functions in $\mathbb{R}^{n-1}$ which are periodic of period $Y$. Let $L^2_{\text{per}}(Y)$ (respectively, $H^1_{\text{per}}(Y)$) be the completion of $C_{\text{per}}(Y)$ in the norm of $L^2(Y)$ (respectively, $H^1(Y)$).

**Definition 4.1.** A sequence of functions $u_\varepsilon \in L^2((0,T) \times G_\varepsilon)$ is said to two-scale converge weakly to $u_0(t, \bar{x}, \bar{y})$ belonging to $L^2((0,T) \times \Sigma \times Z)$ if, for any

$$\psi(t, \bar{x}, \bar{y}, y_n) \in C \left( \left[0,T\right] \times \bar{\Sigma}, C_{\text{per}}([0,1]^{n-1}, C([-1,1])) \right),$$

we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^T \int_{G_\varepsilon} u_\varepsilon(t,x) \psi \left(t, \bar{x}, \frac{x}{\varepsilon} \right) \, dx \, dt = \int_0^T \int_{\Sigma} \int_Z u_0(t, \bar{x}, \bar{y}) \psi(t, \bar{x}, \bar{y}) \, dy \, d\bar{x} \, dt.\tag{4.1}$$

A sequence $u_\varepsilon \in L^2((0,T) \times G_\varepsilon)$ which converges weakly to $u_0 \in L^2((0,T) \times \Sigma \times Z)$ is said to converge strongly in the two-scale sense to the limit $u_0$ if

$$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{\varepsilon}} \left\| u_\varepsilon \right\|_{L^2((0,T) \times G_\varepsilon)} = \left\| u_0 \right\|_{L^2((0,T) \times \Sigma \times Z)}.\tag{4.2}$$

**Remark 1.** If $u_0$ in (4.2) has the property that $u_0(\cdot, \cdot, \bar{z}) \in L^2((0,T) \times G_\varepsilon)$, then the relation (4.2) is equivalent to

$$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{\varepsilon}} \left\| u_\varepsilon(t,x) - u_0 \left(t, \bar{x}, \frac{x}{\varepsilon} \right) \right\|_{L^2((0,T) \times G_\varepsilon)} = 0.$$

A sufficient condition for $u_0(\cdot, \cdot, \bar{z})$ to be in $L^2((0,T) \times G_\varepsilon)$ is, e.g., that $u_0 \in L^2((0,T) \times \Sigma, C_{\text{per}}(Y,C([-1,1])))$. For more details concerning this topic, see [1, Remark 1.10].

The main compactness result obtained for standard two-scale convergence in [12] and [1] can be generalized for the case of sequences defined on thin domains with microstructure.

**Proposition 4.2.** Let $u_\varepsilon$ be a sequence in $L^2((0,T) \times G_\varepsilon)$, such that

$$\frac{1}{\sqrt{\varepsilon}} \left\| u_\varepsilon \right\|_{L^2((0,T) \times G_\varepsilon)} \leq C$$

with a positive constant $C$, independent of $\varepsilon$. Then there exists a subsequence (which we still denote by $\varepsilon$) and a limit function $u_0 \in L^2((0,T) \times \Sigma \times Z)$, such that

$$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{\varepsilon}} \int_0^T \int_{G_\varepsilon} u_\varepsilon(t,x) \varphi \left(t, \bar{x}, \frac{x}{\varepsilon} \right) \, dx \, dt = \int_0^T \int_{\Sigma} \int_Z u_0(t, \bar{x}, \bar{y}) \varphi(t, \bar{x}, \bar{y}) \, dy \, d\bar{x} \, dt \tag{4.3}$$

for every test function $\varphi \in C \left( [0,T] \times \bar{\Sigma}, C_{\text{per}}([0,1]^{n-1}, C([-1,1])) \right)$.

**Proof.** Using Lemma 4.3 below, this result can be proved analogously to Theorem 1.2 in [1]. However, we have to take into account new aspects like time dependence and domains shrinking to a hypersurface.
Let us first consider functions \( u_\varepsilon \), which do not vary with respect to time. Let \( \varphi \in C(\Sigma, C_{\text{per}}(Y, C([-1, 1]))) \) and define
\[
\mu_\varepsilon(\varphi) := \frac{1}{\varepsilon} \int_{G_\varepsilon} u_\varepsilon(x) \varphi \left( \frac{x}{\varepsilon} \right) dx.
\]
Since
\[
|\mu_\varepsilon(\varphi)| \leq \frac{1}{\sqrt{\varepsilon}} ||u_\varepsilon||_{L^2(G_\varepsilon)} \cdot \left\{ \frac{1}{\varepsilon} \int_{G_\varepsilon} \left| \varphi \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon} \right) \right|^2 dx \right\}^{1/2} \leq C ||\varphi||_B,
\]
\( \mu_\varepsilon \) is a bounded sequence of functionals on \( B = C(\Sigma, C_{\text{per}}(Y, C([-1, 1]))) \). Since this space is a separable Banach space, one can extract a subsequence of \( \mu_\varepsilon \) (denoted \( \mu_\varepsilon \) again) which weak*-converges to a limit functional \( \mu_0 \in B \). Using now the boundedness of \( u_\varepsilon \) and Lemma 4.3, we obtain for every \( \varphi \in B \)
\[
|\mu_0(\varphi)|^2 = \lim_{\varepsilon \to 0} |\mu_\varepsilon(\varphi)|^2 \leq C \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{G_\varepsilon} \left| \varphi \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon} \right) \right|^2 dx = C ||\varphi||_{L^2(\Sigma \times Z)}^2.
\]
From the density of \( B \) in \( L^2(\Sigma \times Z) \), it follows that \( \mu_0 \) is a bounded functional on the Hilbert space \( L^2(\Sigma \times Z) \). Thus the Riesz representation theorem implies the existence of a functional \( u_0 \in L^2(\Sigma \times Z) \) such that (4.3) is satisfied.

The proof of the theorem for the case of time-dependent functions can be reduced to the previous one by considering functions defined on the spatial domain with values in the separable Banach space \( L^2((0, T)) \).

**Lemma 4.3.** Let
\[
B = C([0, T] \times \Sigma, C_{\text{per}}(Y, C([-1, 1])))
\]
be the space of continuous functions on \( [0, T] \times \Sigma \) with values in the space \( C_{\text{per}}(Y, C([-1, 1])) \) of continuous functions on \( Z \) and \( Y \) periodic. \( B \) is a separable Banach space, which is dense in \( L^p((0, T) \times \Sigma \times Z) \) for \( 1 \leq p < \infty \), and for every \( \varphi \in B \) the following assertions hold:
\[
\frac{1}{\varepsilon} \int_0^T \int_{G_\varepsilon} \left| \varphi \left( t, \frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right) \right|^p dx dt \leq C ||\varphi||^p_B
\]
and
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^T \int_{G_\varepsilon} \left| \varphi \left( t, \frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right) \right|^p dx dt = \int_0^T \int_{\Sigma} \int_Z \left| \varphi(t, \bar{x}, y) \right|^p dy \bar{d} \bar{x} dt.
\]

The proof of (4.4) is obvious. To prove (4.5), we consider a paving of \( \Sigma \) with \( \varepsilon \)-cells and approximate \( \varphi \) by step functions with respect to the variable \( \bar{x} \in \Sigma \). Using then the periodicity of \( \varphi \) with respect to the variable \( \bar{y} \in Y \) and taking the limit for \( \varepsilon \to 0 \), the assertion follows.

Next we investigate the situation where we also have bounds on the gradients.

**Proposition 4.4.**
(i) Let \( u_\varepsilon \) be a sequence of functions in \( L^2((0, T), H^1(G_\varepsilon)) \), such that
\[
\frac{1}{\sqrt{\varepsilon}} ||u_\varepsilon||_{L^2((0, T) \times G_\varepsilon)} + \frac{1}{\sqrt{\varepsilon}} ||\nabla u_\varepsilon||_{L^2((0, T) \times G_\varepsilon)} \leq C.
\]

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Then there exist functions \( u_0 \in L^2((0,T), H^1(\Sigma)) \) and \( u_1 \in L^2((0,T) \times \Sigma, H_{\text{per}}^1(Y, H^1[-1,1])/\mathbb{R}) \), such that

\[
\begin{align*}
\varepsilon \frac{t}{s} u_0(t, \bar{x}) & \quad \text{weakly in the two-scale sense,} \\
\nabla u_0 \varepsilon \frac{t}{s} - \frac{\partial_y \varepsilon u_0(t, \bar{x})}{\varepsilon} + \nabla_y u_1(t, \bar{x}, y) & \quad \text{weakly in the two-scale sense.}
\end{align*}
\]

(ii) Let \( u_\varepsilon \) be a sequence in \( L^2((0,T), H^1(G_\varepsilon)) \), such that

\[
\frac{1}{\varepsilon} \| u_\varepsilon \|_{L^2(0,T \times G_\varepsilon)} + \sqrt{\varepsilon} \| \nabla u_\varepsilon \|_{L^2(0,T \times G_\varepsilon)} \leq C.
\]

Then there exists \( u_0 \in L^2((0,T) \times \Sigma, H_{\text{per}}^1(Y, H^1[-1,1])) \) such that

\[
\begin{align*}
\varepsilon \frac{t}{s} u_\varepsilon & \quad \text{weakly in the two-scale sense,} \\
\nabla u_\varepsilon \varepsilon \frac{t}{s} - \frac{\partial_y \varepsilon u_0(t, \bar{x})}{\varepsilon} + \nabla_y u_1(t, \bar{x}, y) & \quad \text{weakly in the two-scale sense.}
\end{align*}
\]

The proof of this theorem is given by using Theorem 4.2 and arguments similar to the ones in Proposition 1.14 in [1].

**Equivalent formulation.** When we are dealing with nonlinear problems, the weak two-scale convergence is no longer sufficient for passing to the limit in the nonlinear terms. Here one needs strongly two-scale convergent sequences. However, it is very difficult to show directly the strong two-scale convergence for sequences defined on varying domains, e.g., thin heterogeneous layers. In such cases we use an equivalent characterization of the two-scale convergence described below. This reformulation has the strong advantage that sequences of functions defined on varying domains are transformed to sequences on fixed domains.

Contributions to the development of this method are given in [3], [5], [8], and [9]. In our paper we adapt this method to the case of thin heterogeneous layers and nonlinear problems. Recently, in [16], a proof for the “thick” Neumann sieve was given using multiscale techniques based on [9].

For each \( \varepsilon > 0 \), let us consider the lattice

\[
\mathcal{A}_\varepsilon = \{ \bar{x} = \varepsilon i, i \in \mathbb{Z}^{n-1} \} = \varepsilon \mathbb{Z}^{n-1}.
\]

To every \( \bar{x} \in \Sigma \) we can associate a unique lattice point \( c_\varepsilon(\bar{x}) := \varepsilon \lfloor \bar{x} \rfloor \in \mathcal{A}_\varepsilon \), such that \( \bar{x} \in c_\varepsilon(\bar{x}) + \varepsilon Y \). For simplicity, from now on we consider domains \( G_\varepsilon \) of the form \( G_\varepsilon = [0,1]^{n-1} \times [-\varepsilon, \varepsilon] \).

**Definition 4.5.** We define the operator \( L_\varepsilon \), mapping measurable functions \( u_\varepsilon \) on \((0,T) \times G_\varepsilon \) to measurable functions \( L_\varepsilon u_\varepsilon \) on \((0,T) \times \Sigma \times Z \), with

\[
(L_\varepsilon u_\varepsilon)(t, \bar{x}, y) = \varepsilon u(t, (c_\varepsilon(\bar{x}), 0) + \varepsilon y)
\]

for a.e. \( \bar{x} \in c_\varepsilon(\bar{x}) + \varepsilon Y, (t, y) \in (0,T) \times Z \).

The following properties of the operator \( L_\varepsilon \) can be proved in analogy to Lemma 2 in [3].

**Lemma 4.6.** For \( u_\varepsilon, v_\varepsilon \in L^2((0,T) \times G_\varepsilon) \), we have

\[
\begin{align*}
(L_\varepsilon u_\varepsilon, L_\varepsilon v_\varepsilon)_{L^2((0,T) \times \Sigma \times Z)} &= \frac{1}{\varepsilon} (u_\varepsilon, v_\varepsilon)_{L^2((0,T) \times G_\varepsilon)}, \\
\|L_\varepsilon u_\varepsilon\|_{L^2((0,T) \times \Sigma \times Z)} &= \frac{1}{\sqrt{\varepsilon}} \|u_\varepsilon\|_{L^2((0,T) \times G_\varepsilon)}, \\
\nabla_y L_\varepsilon u_\varepsilon &= \varepsilon L_\varepsilon (\nabla_x u_\varepsilon) \quad \text{a.e. in} \ (0,T) \times \Sigma \times Z.
\end{align*}
\]
The next proposition shows that weak (strong) two-scale convergence for a sequence \( u_\varepsilon \in L^2((0, T) \times G_\varepsilon) \) is equivalent to weak (strong) convergence for the sequence \( L_\varepsilon u_\varepsilon \) in \( L^2((0, T) \times \Sigma \times Z) \).

**Proposition 4.7.** Let \( u_\varepsilon \in L^2((0, T) \times G_\varepsilon) \) be a sequence, such that

\[
\frac{1}{\varepsilon} ||u_\varepsilon||_{L^2((0,T) \times G_\varepsilon)} \leq C.
\]

Then there exists a subsequence (again denoted by \( u_\varepsilon \)) and a limit function \( u_0 \in L^2((0, T) \times \Sigma \times Z) \), such that the following statements are equivalent:

(i) \( u_\varepsilon \xrightarrow{\text{t.s.}} u_0 \) weakly (strongly) in the two-scale sense.
(ii) \( L_\varepsilon u_\varepsilon \xrightarrow{\text{t.s.}} u_0 \) weakly (strongly) in \( L^2((0, T) \times \Sigma \times Z) \).

**Proof.** Since by Lemma 4.6

\[
||L_\varepsilon u_\varepsilon||_{L^2((0,T) \times \Sigma \times Z)} = \frac{1}{\varepsilon} ||u_\varepsilon||_{L^2((0,T) \times G_\varepsilon)} \leq C
\]

it follows that there exists a subsequence of \( u_\varepsilon \) (again denoted by \( u_\varepsilon \)), and there exist \( u_0, u_* \in L^2((0, T) \times \Sigma \times Z) \), such that

\[
u_\varepsilon \xrightarrow{\text{t.s.}} u_0 \quad \text{weakly in the two-scale sense},
\]

\[
L_\varepsilon u_\varepsilon \xrightarrow{\text{t.s.}} u_* \quad \text{weakly in } L^2((0, T) \times \Sigma \times Z).
\]

Then a proof analogous to that of Proposition 4.6 in [5] shows that \( u_0 \equiv u_* \). To prove the equivalence of statements (i) and (ii) with respect to the strong convergences let us remark that

\[
||L_\varepsilon u_\varepsilon - u_0||_{L^2((0,T) \times \Sigma \times Z)} = \int_0^T \int_{\Sigma} \int_Z |L_\varepsilon u_\varepsilon - u_0|^2 \, dy \, d\varepsilon \, dt
\]

\[
= ||L_\varepsilon u_\varepsilon||_{L^2((0,T) \times \Sigma \times Z)}^2 - 2 \int_0^T \int_{\Sigma} \int_Z (L_\varepsilon u_\varepsilon)u_0 \, dy \, d\varepsilon \, dt + ||u_0||_{L^2((0,T) \times \Sigma \times Z)}^2
\]

\[
= \frac{1}{\varepsilon} ||u_\varepsilon||_{L^2((0,T) \times G_\varepsilon)}^2 - 2 \int_0^T \int_{\Sigma} \int_Z (L_\varepsilon u_\varepsilon)u_0 \, dy \, d\varepsilon \, dt + ||u_0||_{L^2((0,T) \times \Sigma \times Z)}^2.
\]

Taking now the limits \( \varepsilon \to 0 \) on both sides of this equality, the equivalence of (i) and (ii) is proved. \( \Box \)

**5. Proofs for the convergence results stated in Propositions 2.1 and 2.2 and Theorem 2.3.** From the a priori estimates we see that we have different compactness properties for the solutions \( u_\varepsilon \) on the subdomains \( \Omega^\pm_\varepsilon \) and \( \Omega^M_\varepsilon \). Thus we have to study the convergence of the sequences \( u^\pm_\varepsilon \) and \( u^M_\varepsilon \) separately.

**5.1. Convergence in the bulk.** In this subsection, we give the proof of the first three convergence results from Proposition 2.1.

**Proof.** Let us consider the transformations

\[
\Omega^\pm_\varepsilon \mapsto \Omega^\pm_\varepsilon, \quad (\bar{x}, \bar{x}_n) \mapsto (\bar{x}, x_n) = \left( \bar{x}, \frac{H}{H} \bar{x}_n \pm \varepsilon \right)
\]
and define

\begin{equation}
\tilde{u}^\pm_{je} : [0, T] \times \Omega^\pm \to \mathbb{R}^m, \quad \tilde{u}^\pm_{je}(t, \bar{x}, \bar{x}_n) = u^\pm_e \left( t, \bar{x}, \frac{H - \varepsilon}{H} \bar{x}_n \pm \varepsilon \right).
\end{equation}

Using the transformation formula for integrals, we can easily show the estimates for the functions \( \tilde{u}^\pm_{je} \),

\begin{equation}
||\tilde{u}^\pm_{je}||_{L^2((0, T; H^1(\Omega^\pm))} \leq C ||u^\pm_{je}||_{L^2((0, T; H^1(\Omega^\pm))},
\end{equation}

\begin{equation}
||\partial_t \tilde{u}^\pm_{je}||_{L^2((0, T; L^2(\Omega^\pm))} \leq C ||\partial_t u^\pm_{je}||_{L^2((0, T; L^2(\Omega^\pm))},
\end{equation}

with a constant \( C \) independent of \( \varepsilon \). Now, since the functions \( \tilde{u}^\pm_{je} \) are defined on fixed domains \( \Omega^\pm \), standard compactness results together with the estimates (5.3) and (5.4) imply that there exist \( u^\pm_{0} \in L^2((0, T), H^1(\Omega^\pm, \mathbb{R}^m)) \) with \( \partial_t u^\pm_{0} \in L^2((0, T), L^2(\Omega^\pm, \mathbb{R}^m)) \), such that up to a subsequence

\begin{align*}
\tilde{u}^\pm_{je} & \to u^\pm_{0} \quad \text{weakly in } L^2((0, T), H^1(\Omega^\pm)), \\
\partial_t \tilde{u}^\pm_{je} & \to \partial_t u^\pm_{0} \quad \text{weakly in } L^2((0, T), L^2(\Omega^\pm)), \\
\tilde{u}^\pm_{je} & \to u^\pm_{0} \quad \text{strongly in } L^2((0, T), L^2(\Omega^\pm)).
\end{align*}

The strong convergence follows from the estimate

\begin{equation}
||\tilde{u}^\pm_{je}||_{L^2((0, T; H^1(\Omega^\pm))} + ||\partial_t \tilde{u}^\pm_{je}||_{L^2((0, T; L^2(\Omega^\pm))} \leq C
\end{equation}

and a compactness theorem of Lions; see [10, Theorem 1, p. 58].

Now let \( \varphi \in C_0^\infty((0, T) \times \Omega^\pm, \mathbb{R}^m) \). Using the transformations (5.1), (5.2), we have

\begin{align*}
\int_{0}^{T} \int_{\Omega^\pm} (\chi_{\Omega^\pm} u_{je})(t, x) \varphi(t, x) \, dx \, dt &= \int_{0}^{T} \int_{\Omega^\pm} u_{je}^\pm(t, \bar{x}, \bar{x}_3) \varphi(t, \bar{x}, \bar{x}_3(\bar{x}) \, d\bar{x}_3 \, dt \\
&= \frac{H - \varepsilon}{H} \int_{0}^{T} \int_{\Omega^\pm} \tilde{u}_{je}^\pm(t, \bar{x}, \bar{x}_3) \varphi(t, \bar{x}, \bar{x}_3) \, d\bar{x}_3 \, dt \\
&\quad + \frac{H - \varepsilon}{H} \int_{0}^{T} \int_{\Omega^\pm} \tilde{u}_{je}^\pm(t, \bar{x}, \bar{x}_3) \left[ \varphi(t, \bar{x}, \bar{x}_3) - \varphi(t, \bar{x}, \bar{x}_3) \right] \, d\bar{x}_3 \, dt \\
&\quad - \int_{0}^{T} \int_{\Omega^\pm} u_{0j}^\pm(t, \bar{x}, \bar{x}) \varphi(t, \bar{x}, \bar{x}) \, d\bar{x}_3 \, dt
\end{align*}

due to the convergence properties of \( \tilde{u}^\pm_{je} \) and the smoothness of \( \varphi \). Thus

\begin{equation}
\chi_{\Omega^\pm} u_{je} \to u_{0j}^\pm \quad \text{weakly in } L^2((0, T), L^2(\Omega^\pm)).
\end{equation}

Additionally, we have that

\begin{equation}
||\chi_{\Omega^\pm} u_{je}||_{L^2((0, T), L^2(\Omega^\pm))} = \sqrt{1 - \varepsilon/H} ||\tilde{u}^\pm_{je}||_{L^2((0, T; L^2(\Omega^\pm))} \to ||u^\pm_{0j}||_{L^2((0, T; L^2(\Omega^\pm))},
\end{equation}

since \( \tilde{u}^\pm_{je} \) converges to \( u^\pm_{0j} \) strongly in \( L^2((0, T) \times \Omega^\pm, \mathbb{R}^m) \). Thus, statement 1 of Proposition 2.1 is proved. The proofs of statements 2 and 3 follow along the same line. \( \Box \)
5.2. Convergence for the traces on the interfaces bulk/layer. The compactness of the traces of \( u_\varepsilon \) on \( S^\pm_\varepsilon \) is crucial for the control of the solutions in the layer. In the following, we give the proof of Proposition 2.2.

Proof. Since
\[
\|\tilde{u}^\pm_{j\varepsilon}\|_{L^2((0,T),H^1(\Omega^\pm))} + \|\partial_t \tilde{u}^\pm_{j\varepsilon}\|_{L^2((0,T),L^2(\Omega^\pm))} \leq C
\]
and the embedding
\[
H^1(\Omega^\pm) \hookrightarrow H^\beta(\Omega^\pm)
\]
is compact for every \( \frac{1}{2} < \beta < 1 \), it follows from Lions compactness theorem [10, Theorem 1, p. 58] that there exists a subsequence such that
\[
\tilde{u}^\pm_{j\varepsilon} \to u^\pm_{j0} \quad \text{strongly in} \quad L^2((0,T),H^\beta(\Omega^\pm)).
\]
Due to the continuity of the embedding
\[
H^\beta(\Omega^\pm) \hookrightarrow L^2(\partial\Omega^\pm) \quad \text{for} \quad \frac{1}{2} < \beta < 1,
\]
it follows that
\[
\|\tilde{u}^\pm_{j\varepsilon} - u^\pm_{j0}\|_{L^2((0,T)\times\Sigma)} \leq C\|\tilde{u}^\pm_{j\varepsilon} - u^\pm_{j0}\|_{L^2((0,T),H^\beta(\Omega^\pm))} \to 0 \quad \text{for} \quad \varepsilon \to 0.
\]
Thus the first assertion is proved. To prove the second one, we notice that, since \( \tilde{u}_\varepsilon|_{\Sigma} \) is strongly convergent in \( L^2((0,T)\times\Sigma,\mathbb{R}^m) \), it is also weakly two-scale convergent to the same limit; see [1]. Then, using again (5.1) and (5.2), we obtain for \( \varepsilon \to 0 \)
\[
\int_0^T \int_{S^\pm_\varepsilon} u^\pm_{j\varepsilon}(t,x) \varphi_j(t,\tilde{x},\frac{x}{\varepsilon}) \, dx \, dt = \frac{H - \varepsilon}{H} \int_0^T \int_{\Sigma} \tilde{u}^\pm_{j\varepsilon}(t,\tilde{x},0) \varphi_j(t,\tilde{x},\frac{\tilde{x}}{\varepsilon},\pm 1) \, d\tilde{x} \, dt
\]
\[
\quad \to \int_0^T \int_{\Sigma} \int_{Y} u^\pm_{j0}(t,\tilde{x},0) \varphi_j(t,\tilde{x},\tilde{y},\pm 1) \, d\tilde{y} \, d\tilde{x} \, dt. \quad \Box
\]

5.3. Weak two-scale convergence in the layer. The compactness results with respect to the weak two-scale convergence of \( u^M_\varepsilon \) in the layer stated in statements 4–6 of Proposition 2.1 follow directly from Proposition 4.4 and the a priori estimates for \( u^M_\varepsilon \). It remains to prove the coupling condition between the effective solutions in the bulk regions and the local solution in the layer given in (2.19).

Proof. Let us start from the identity
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^T \int_{\Omega^+_\varepsilon} \varepsilon \nabla u^M_{j\varepsilon}(t,x) \varphi_j(t,\tilde{x},\frac{x}{\varepsilon}) \, dx \, dt = \int_0^T \int_{\Sigma} \int_{Y} \nabla y u^M_{j0}(t,\tilde{x},y) \varphi_j(t,\tilde{x},y) \, dy \, d\tilde{x} \, dt
\]
for any \( \varphi_j \in C^\infty([0,T] \times \Sigma \times Z,\mathbb{R}^n) \) periodic in \( Y = [0,1]^{n-1} \) and with compact support with respect to \( \tilde{x} \in \Sigma \). Integrating by parts on the left-hand side and using the continuity of the solution \( u_\varepsilon \) on \( S^+_\varepsilon \), respectively, \( S^-_\varepsilon \), and Proposition 2.2, we
obtain

\[
\lim_{\varepsilon \to 0} \left\{ -\frac{1}{\varepsilon} \int_0^T \int_{\Omega^M_\varepsilon} u^M_{j\varepsilon}(t, x) \left( \varepsilon \nabla_x \varphi_j \left( t, \bar{x}, \frac{x}{\varepsilon} \right) + \nabla_y \varphi_j \left( t, \bar{x}, \frac{x}{\varepsilon} \right) \right) \, dx \, dt \\
+ \int_0^T \int_{S^+_\varepsilon} u_{j\varepsilon}^+(t, x) \varphi_j \left( t, \bar{x}, \frac{x}{\varepsilon} \right) \cdot \nu \, dx \, dt \\
+ \int_0^T \int_{S^-_\varepsilon} u_{j\varepsilon}^-(t, x) \varphi_j \left( t, \bar{x}, \frac{x}{\varepsilon} \right) \cdot \nu \, dx \, dt \right\} = - \int_0^T \int_{\Sigma} u^M_{j0}(t, \bar{x}, y) \nabla_y \varphi_j(t, \bar{x}, y) \, dy \, d\bar{x} \, dt \\
+ \int_0^T \int_{\Sigma} u_{j0}^+(t, \bar{x}, 0) \varphi_j(t, \bar{x}, \bar{y}, 0) \cdot n \, dy \, d\bar{x} \, dt \\
- \int_0^T \int_{\Sigma} u_{j0}^-(t, \bar{x}, 0) \varphi_j(t, \bar{x}, \bar{y}, -1) \cdot n \, dy \, d\bar{x} \, dt.
\]

Here \( n = (0, \ldots, 0, 1) \). By equality between the two limits, we obtain statement (2.19) of the theorem. \( \Box \)

**5.4. Strong two-scale convergence in the layer.** In the following, we prove strong two-scale convergence for the sequence \( u^M_{\varepsilon} \). The method to be used is of general interest. Here, we are in the situation that the standard estimates of the solutions \( u^M_{\varepsilon} \) imply only weak two-scale convergence, and due to the scaling in the diffusion coefficients we cannot use the method of bounded extensions to get strong convergence for \( u^M_{\varepsilon} \) in \( L^2 \). The best compactness result we get for \( u^M_{\varepsilon} \) is strong two-scale convergence.

To prove this, we show the strong convergence of the transformed solution \( L^M_{\nu^M} \) in \( L^2((0, T) \times \Sigma \times Z) \) (see Proposition 4.7). The control of the dependence of \( L^M_{\nu^M} \) on \( y \) and \( t \) is standard since we can use the differential equations in the cells. However, the dependence on \( x \) poses more serious problems since the functions \( L^M_{\nu^M} \) are only step functions with respect to \( x \). To get equicontinuity in \( L^2 \) also with respect to shifts in \( x \), we mainly have to compare solutions of the differential equations in different cells. A similar argument, although in a quite different situation, can be found in [7], which deals with the Neumann problem for rapidly oscillatory boundaries.

Verifying the compactness criteria requires the extension of \( u^M_{\varepsilon} \) into a neighborhood of \( \Omega^M_{\varepsilon} \). We construct the extension

\[
\hat{u}^M_{\varepsilon} : (0, T) \times \mathbb{R}^{n-1} \times ]-\varepsilon, \varepsilon[ \to \mathbb{R}^m
\]
as follows.

First, we extend \( u^M_{\varepsilon} \) by reflection with respect to the plane \( \{(x_1, x_2, \ldots, x_n) : x_1 = 0\} \) to

\[
\hat{u}^M_{\varepsilon}(t, x) = \begin{cases} u^M_{\varepsilon}(t, x), & (t, x) \in (0, T) \times \Omega^M_{\varepsilon} \\
u^M_{\varepsilon}(t, -x_1, x_2, \ldots, x_{n-1}, x_n), & (t, -x_1, x_2, \ldots, x_{n-1}, x_n) \in (0, T) \times \Omega^M_{\varepsilon}.
\end{cases}
\]

Then we repeat this extension procedure with respect to the planes \( \{(x_1, x_2, \ldots, x_n) : x_2 = 0\}, \ldots, \{(x_1, x_2, \ldots, x_n) : x_{n-1} = 0\} \) and obtain the extension \( \hat{u}^M_{\varepsilon} \) on the domain

\[
(0, T) \times \hat{\Omega}^M_{\varepsilon} = (0, T) \times ]-1, 1[^{n-1} \times ]-\varepsilon, \varepsilon[.
\]

Now, we further extend \( \hat{u}^M_{\varepsilon} \) by periodicity to \( (0, T) \times \mathbb{R}^{n-1} \times ]-\varepsilon, \varepsilon[ \). Due to our extension procedure, we have that

\[
\hat{u}^M_{\varepsilon} \in L^2((0, T), H^1_{\text{per}}(-1, 1[^{n-1}, H^1([-\varepsilon, \varepsilon[; \mathbb{R}^m)))
\]
and
\[ \| \hat{u}_\varepsilon^M \|_{L^2((0,T),H^1(\Omega^M,\mathbb{R}^m))} \leq C \| u_\varepsilon^M \|_{L^2((0,T),H^1(\Omega^M,\mathbb{R}^m))}. \]

In an analogous way, we extend \( u_\varepsilon^\pm \) to \( \hat{u}_\varepsilon^\pm \), satisfying
\[ \hat{u}_\varepsilon^+ \in L^2((0,T),H^1_{\text{per}}([-1,1]^{n-1},H^1(\varepsilon,\mathbb{R}^m))), \]
\[ \hat{u}_\varepsilon^- \in L^2((0,T),H^1_{\text{per}}([-1,1]^{n-1},H^1([-\varepsilon,\varepsilon]\times\mathbb{R}^m))). \]

We also define
\[ \hat{S}_\varepsilon^\pm = \{ x = (\bar{x},x_n) : \bar{x} \in [-1,1]^{n-1}, x_n = \pm \varepsilon \} \]
and remark that the traces \( \hat{u}_\varepsilon^\pm |_{\hat{S}_\varepsilon^\pm} \) are periodic with respect to \( \bar{x} \) with period \([-1,1]^{n-1}\), and the continuity condition
\[ \hat{u}_\varepsilon^M |_{\hat{S}_\varepsilon^\pm} = \hat{u}_\varepsilon^M |_{\hat{S}_\varepsilon^\pm} \]
holds. Let us now give the proof of Theorem 2.3.

**Proof.** We denote
\[ \delta \hat{u}_\varepsilon^M (t,x) = \hat{u}_\varepsilon^M (t,x + (t,0)\varepsilon) - \hat{u}_\varepsilon^M (t,x), \]
\[ \delta \hat{u}_\varepsilon^\pm (t,x) = \hat{u}_\varepsilon^\pm (t,x + (t,0)\varepsilon) - \hat{u}_\varepsilon^\pm (t,x), \]
\[ \delta g(t,x) = g \left( \frac{x}{\varepsilon}, u_\varepsilon^M (t,x + (t,0)\varepsilon) \right) - g \left( \frac{x}{\varepsilon}, u_\varepsilon^M (t,x) \right) \]
\[ \delta U_0^M (x) = U_0^M \left( \frac{x}{\varepsilon} + \varepsilon \right) - U_0^M \left( \frac{x}{\varepsilon} \right). \]

Now let \( \delta v_\varepsilon = (\delta v_{1\varepsilon}, \ldots, \delta v_{m\varepsilon}) \) be the solution to the following problem:
\[ \frac{1}{\varepsilon} \partial_t (\delta v_{j\varepsilon})(t,x) - \varepsilon \nabla \left( D_j^M \left( \frac{x}{\varepsilon} \right) \nabla (\delta v_{j\varepsilon})(t,x) \right) = 0 \quad \text{in } (0,T) \times \hat{\Omega}_\varepsilon^M, \]
\[ \delta v_{j\varepsilon}(t,x) = \delta \hat{u}_\varepsilon^\pm (t,x) \quad \text{on } (0,T) \times \hat{S}_\varepsilon^\pm, \]
\[ \delta v_{j\varepsilon}(0,x) = 0 \quad \text{in } \hat{\Omega}_\varepsilon^M. \]

The following estimates hold:

(5.5) \[ \frac{1}{\varepsilon} \| \delta v_{j\varepsilon} \|_{L^\infty((0,T),L^2(\hat{\Omega}_\varepsilon^M,\mathbb{R}^m))} + \sqrt{\varepsilon} \| \nabla (\delta v_{j\varepsilon}) \|_{L^2((0,T) \times \hat{\Omega}_\varepsilon^M,\mathbb{R}^m)} \leq C, \]

(5.6) \[ \frac{1}{\varepsilon} \| \delta v_{j\varepsilon} \|_{L^2((0,T) \times \hat{\Omega}_\varepsilon^M,\mathbb{R}^m)} \leq C(\| \delta \hat{u}_\varepsilon^+ \|_{L^2((0,T) \times \hat{S}_\varepsilon^+,\mathbb{R}^m)} + \| \delta \hat{u}_\varepsilon^- \|_{L^2((0,T) \times \hat{S}_\varepsilon^-,\mathbb{R}^m)}). \]

To prove (5.5), we test the equation for \( \delta v_{j\varepsilon} \) with \( \delta v_{j\varepsilon} - \delta \hat{u}_\varepsilon^M \) and use the same techniques as in the first part the proof of Lemma 3.1, together with the a priori estimates for \( u_\varepsilon^M \). For the proof of (5.6), we use the solution \( h = (h_1, \ldots, h_m) \) to the adjoint problem
\[ -\frac{1}{\varepsilon} \partial_t h_j(t,x) - \varepsilon \nabla (D_j^M \left( \frac{x}{\varepsilon} \right) \nabla h_j(t,x)) = \frac{1}{\varepsilon} \delta v_{j\varepsilon}(t,x) \quad \text{in } (0,T) \times \hat{\Omega}_\varepsilon^M, \]
\[ h(t,x) = 0 \quad \text{on } (0,T) \times \hat{S}_\varepsilon^+, \]
\[ h(T,x) = 0 \quad \text{in } \hat{\Omega}_\varepsilon^M, \]
\[ h \text{ is periodic with period } [-1,1]^{n-1}. \]
Regularity theory for parabolic systems together with a scaling argument implies that 
\( h \in L^2((0,T), H^2(\Omega^M_{\varepsilon}, \mathbb{R}^m)) \) and we have

\[
(5.7) \quad \frac{1}{\varepsilon^2} \| h_j \|_{L^2((0,T) \times \Omega^M_{\varepsilon})} + \sqrt{\varepsilon} \| \nabla h_j \|_{L^2((0,T) \times \Omega^M_{\varepsilon}, \mathbb{R}^n)} \\
+ \varepsilon \sqrt{\varepsilon} \| \nabla^2 h_j \|_{L^2((0,T) \times \Omega^M_{\varepsilon}, \mathbb{R}^{n^2})} \leq C \varepsilon \| \delta v_{je} \|_{L^2((0,T) \times \Omega^M_{\varepsilon}, \mathbb{R}^m)}.
\]

Let us now multiply the equation for \( h \) by \( \delta v_{je} \) and integrate. We obtain

\[
\frac{1}{\varepsilon} \int_0^T \int_{\Omega^M_{\varepsilon}} |\delta v_{je}|^2 dx dt \\
= \int_0^T \int_{\Omega^M_{\varepsilon}} \left( -\frac{1}{\varepsilon} \partial_t h_j - \varepsilon \nabla \left( D^M_{\varepsilon} \left( \frac{x}{\varepsilon} \right) \nabla h_j \right) \right) (\delta v_{je}) \, dx dt \\
= \int_0^T \int_{\Omega^M_{\varepsilon}} \frac{1}{\varepsilon} h_j \partial_t (\delta v_{je}) + \varepsilon D^M_{\varepsilon} \left( \frac{x}{\varepsilon} \right) \nabla h_j \nabla (\delta v_{je}) dx dt \\
- \varepsilon \int_0^T \int_{S_\varepsilon^+} D^M_{\varepsilon} \left( \frac{x}{\varepsilon} \right) \delta v_{je} \nabla h_j \cdot \nu dx dt - \varepsilon \int_0^T \int_{S_\varepsilon^-} D^M_{\varepsilon} \left( \frac{x}{\varepsilon} \right) \delta v_{je} \nabla h_j \cdot \nu dx dt.
\]

Now, using the problem for \( \delta v_{ce} \) and the fact that \( h \) is equal to zero on \( (0,T) \times \hat{S}_\varepsilon^\pm \), we obtain

\[
\frac{1}{\varepsilon} \| \delta v_{je} \|_{L^2((0,T) \times \Omega^M_{\varepsilon})}^2 \leq C \varepsilon \| \delta v_{je} \|_{L^2((0,T) \times \hat{S}_\varepsilon^+)} \| \nabla h_j \cdot \nu \|_{L^2((0,T) \times \hat{S}_\varepsilon^+)} \\
+ C \varepsilon \| \delta v_{je} \|_{L^2((0,T) \times \hat{S}_\varepsilon^-)} \| \nabla h_j \cdot \nu \|_{L^2((0,T) \times \hat{S}_\varepsilon^-)} \\
\leq C \varepsilon \left( 1 + \sqrt{\varepsilon} \| \nabla h_j \cdot \nu \|_{L^2((0,T) \times \Omega^M_{\varepsilon}, \mathbb{R}^n)} + \sqrt{\varepsilon} \| \nabla (\nabla h_j \cdot \nu) \|_{L^2((0,T) \times \Omega^M_{\varepsilon}, \mathbb{R}^{n^2})} \right) \\
\times \| \delta v_{je} \|_{L^2((0,T) \times \hat{S}_\varepsilon^+) + \| \delta v_{je} \|_{L^2((0,T) \times \hat{S}_\varepsilon^-)}).
\]

For the last inequality, we used the trace estimate for the thin domain \( \Omega^M_{\varepsilon} \) given in Lemma 5.1. The estimate (5.7) and the boundary condition for \( \delta v_{ce} \) on \( \hat{S}_\varepsilon^\pm \) imply the assertion (5.6).

Let us now consider the function \( \delta w_{\varepsilon} = (\delta w_{1\varepsilon}, \ldots, \delta w_{m\varepsilon}) \) defined by

\[
\delta w_{\varepsilon}(t,x) = \delta U^M_{\varepsilon}(t,x) - \delta v_{\varepsilon}(t,x).
\]

In order to avoid the boundary values of \( \delta w_{\varepsilon} \) on the lateral boundary of \( \Omega^M_{\varepsilon} \), we will cut off this part of the boundary and estimate \( \frac{1}{\varepsilon} \| \delta w_{\varepsilon} \|_{L^2((0,T) \times \Omega^M_{\varepsilon}, \mathbb{R}^m)} \), where

\[
\Omega^M_{2\varepsilon} = \{ x \in \Omega^M_{\varepsilon} : 2\varepsilon < x_i < 1 - 2\varepsilon, i = 1, \ldots, n - 1 \}.
\]

Defining \( \Omega^M_{\varepsilon} \) and \( S^\pm_{\varepsilon} \) analogously to (5.8), we have that \( \delta w_{\varepsilon} \) satisfies

\[
\frac{1}{\varepsilon} \partial_t (\delta w_{je})(t,x) - \varepsilon \nabla \left( D^M_{\varepsilon} \left( \frac{x}{\varepsilon} \right) \nabla (\delta w_{je})(t,x) \right) = \frac{1}{\varepsilon} \delta g_j(t,x) \quad \text{in} \ (0,T) \times \Omega^M_{\varepsilon}, \\
(\delta w_{\varepsilon})(t,x) = 0 \quad \text{on} \ (0,T) \times S^\pm_{\varepsilon}, \\
(\delta w_{\varepsilon})(0,x) = \delta U^M_{\varepsilon}(x) \quad \text{in} \ \Omega^M_{\varepsilon}.
\]
Now, we test the equation for $\delta w_\varepsilon$ with a function which vanishes on the lateral boundary of $\Omega^M$. We consider the following cut-off function $\varphi \in C^\infty_0((h, 1-h)^{n-1}, C^\infty([-\varepsilon, \varepsilon[, [0, 1]))$ with the properties

$$\varphi \equiv 1 \quad \text{in } \Omega^M_{2h}, \quad \|\nabla \varphi\|_{L^\infty(\Omega^M_h)} \leq \frac{C}{h}.$$ 

Multiplying the equation for $\delta w_{j\varepsilon}$ by $\delta w_{j\varepsilon} \varphi^2$ and integrating over $\Omega^M_h$, we obtain

\begin{align}
(5.9) \quad & \frac{1}{\varepsilon} \int_{\Omega^M_h} \partial_t (\delta w_{j\varepsilon}) \delta w_{j\varepsilon} \varphi^2 + \varepsilon \int_{\Omega^M_h} D_j \left( \frac{x}{\varepsilon} \right) \nabla (\delta w_{j\varepsilon}) \nabla (\delta w_{j\varepsilon}) \varphi^2 \\
(5.10) \quad & + \varepsilon \int_{\Omega^M_h} D_j \left( \frac{x}{\varepsilon} \right) \nabla (\delta w_{j\varepsilon}) \delta w_{j\varepsilon} 2 \varphi \nabla \varphi = \frac{1}{\varepsilon} \int_{\Omega^M_h} \delta g_j \delta w_{j\varepsilon} \varphi^2.
\end{align}

Integration with respect to time and the Lipschitz continuity of $\varphi$ in the second argument imply

\begin{align}
(5.11) \quad & \frac{1}{\varepsilon} \int_{\Omega^M_h} |\delta w_{j\varepsilon}(t)|^2 \varphi^2 + \varepsilon \int_{\Omega^M_h} \nabla (\delta w_{j\varepsilon})^2 \varphi^2 \\
& \leq C \varepsilon \left( \int_0^t \int_{\Omega^M_h} |\delta w_{j\varepsilon}|^2 \varphi^2 + \int_{\Omega^M_h} |\delta U_{j\varepsilon}^M|^2 \varphi^2 \right) \\
& + C \left( \eta \varepsilon \int_0^t \int_{\Omega^M_h} |\nabla (\delta w_{j\varepsilon})|^2 \varphi^2 + \frac{1}{\eta} \varepsilon \int_0^t \int_{\Omega^M_h} |\delta w_{j\varepsilon}|^2 |\nabla \varphi|^2 \right).
\end{align}

Let us now estimate the last term on the right-hand side of (5.11). Since $\varphi \equiv 1$ in $\Omega^M_{2h}$, the support of $\nabla \varphi$ is contained in the domain

$$T^M_h := \Omega^M_h \setminus \Omega^M_{2h}.$$ 

Using the estimate for $\nabla \varphi$ and Poincaré’s inequality (since $\delta w_{\varepsilon}|_{S^T}\varepsilon = 0$), we have

\begin{align}
(5.12) \quad & \varepsilon \int_0^t \int_{T^M_h} |\delta w_{j\varepsilon}|^2 |\nabla \varphi|^2 \leq C \frac{\varepsilon}{h^2} \int_0^t \int_{T^M_h} |\delta w_{j\varepsilon}|^2 \\
& \leq C \frac{\varepsilon}{h^2} \int_0^t \int_{T^M_h} |\nabla (\delta w_{j\varepsilon})|^2 \varphi^2 \leq C \frac{\varepsilon^3}{h^2} \int_0^t \int_{T^M_h} |\nabla (\delta w_{j\varepsilon})|^2 + |\nabla (\delta v_{j\varepsilon})|^2 \varphi^2 \\
& \leq C \varepsilon^2 \frac{h^2}{h^2}.
\end{align}

For the last inequality, we used the a priori estimates for $u_{j\varepsilon}^M$ and the estimate (5.5) for $\delta v_{j\varepsilon}$.

Going back to relation (5.11), taking $\eta$ small enough, and using the estimate (5.12) and the fact that $\delta w_{j\varepsilon}^M = \delta v_{j\varepsilon} + \delta w_{\varepsilon}$, we get

\begin{align}
(5.13) \quad & \frac{1}{\varepsilon} \int_{\Omega^M_h} |\delta w_{j\varepsilon}(t)|^2 \varphi^2 + \varepsilon \int_0^t \int_{\Omega^M_h} |\nabla (\delta w_{j\varepsilon})|^2 \varphi^2 \\
& \leq C \varepsilon \left( \int_0^t \int_{\Omega^M_h} (|\delta w_{j\varepsilon}^M|^2 + |\delta w_{j\varepsilon}|^2) \varphi^2 + \int_{\Omega^M_h} |\delta U_{j\varepsilon}^M|^2 \varphi^2 \right) + C \varepsilon^2 \frac{h^2}{h^2} \\
& \leq C \varepsilon \left( \int_0^t \int_{\Omega^M_h} (|\delta v_{j\varepsilon}|^2 + |\delta w_{j\varepsilon}|^2) \varphi^2 + \int_{\Omega^M_h} |\delta U_{j\varepsilon}^M|^2 \varphi^2 \right) + C \varepsilon^2 \frac{h^2}{h^2}.
\end{align}
Finally, Gronwall’s lemma implies the estimate

\[
(5.14) \quad \frac{1}{\varepsilon} \| \delta u_{\varepsilon} \|_{L^2((0,T) \times \Omega^M_{nh} \setminus \Omega^M_{2h})}^2 \leq \frac{C}{\varepsilon} \| \delta u_{\varepsilon} \|_{L^2((0,T) \times \Omega^M_{nh})}^2 + \frac{C}{\varepsilon} \| \delta u_{\varepsilon} \|_{L^2((0,T) \times \Omega^M_{nh})}^2 + \frac{C \varepsilon^2}{h^2}.
\]

Now, it remains to estimate \( \frac{1}{\varepsilon^2} \| \delta u_{\varepsilon} \|_{L^2((0,T) \times (\Omega^M \setminus \Omega^M_{2h}), \mathbb{R}^m)} \). For this estimate, we will exploit the fact that the domain \( \Omega^M \setminus \Omega^M_{2h} \) can be decomposed in subdomains which have thickness \( O(h) \) at least in one space direction. Using the \( L^\infty \)-estimate (3.4), we obtain

\[
(5.15) \quad \frac{1}{\varepsilon^2} \left\{ \int_0^1 \int_{\Omega^M \setminus \Omega^M_{2h}} \| u_{\varepsilon} \|_{L^2}^2 \right\} \leq \frac{M^2 e^{AT}}{\sqrt{\varepsilon}} \left\{ \int_0^1 \int_{\Omega^M \setminus \Omega^M_{2h}} dxdt \right\} \leq Ch^{\frac{3}{2}}.
\]

The estimates (5.6), (5.14), and (5.15) imply the estimate (2.22) from the first part of the theorem.

To prove the strong two-scale convergence for \( u_{\varepsilon} \), we will show that

\[
L\varepsilon u_{\varepsilon} \rightarrow u_0 \quad \text{strongly in} \quad L^2((0,T) \times \Sigma \times Z, \mathbb{R}^m).
\]

We first show that for all \( \rho > 0 \), there exists \( \delta > 0 \), such that for all \( \varepsilon \leq \varepsilon_0 \)

\[
(5.16) \quad ||L\varepsilon \tilde{u}_{\varepsilon}^M(t, \bar{x} + \bar{\xi}, y) - L\varepsilon \tilde{u}_{\varepsilon}^M(t, \bar{x}, y)||_{L^2(\Sigma \times Z, \mathbb{R}^m)} < \rho
\]

for all \( \bar{\xi} \in \mathbb{R}^{n-1}, |\bar{\xi}| < \delta \).

Let \( I \subset \mathbb{Z}^{n-1} \), such that

\[
\Sigma = \sum_{i \in I} \varepsilon (Y + i) =: \sum_{i \in I} \varepsilon Y_i.
\]

Obviously, for \( \bar{x} \in \varepsilon Y_i \) we have that \( \left\lfloor \frac{\bar{x}}{\varepsilon} \right\rfloor = i \). For every \( i \in I \) we divide the cell \( \varepsilon Y_i \) into subsets \( \varepsilon Y_i^k \) with \( k \in \{0,1\}^{n-1} \), defined as follows:

\[
\varepsilon Y_i^k = \left\{ \bar{x} \in \varepsilon Y_i, \left[ \frac{\bar{x} + \{\bar{\xi}/\varepsilon\} \varepsilon}{\varepsilon} \right] = \varepsilon (i + k) \right\}.
\]

Then \( \varepsilon Y_i = \bigcup_{k \in \{0,1\}^{n-1}} \varepsilon Y_i^k \). In Figure 5.1, we sketch the subsets \( \varepsilon Y_i^k \) of \( \varepsilon Y_i \) in the case \( n = 3, Y = [0,1]^2 \) and the translation \( \bar{\xi} \) is of the form \( \bar{\xi} = (\xi_1, 0), \xi_1 > 0 \).

Now, let us calculate

\[
\left| \left| \int_0^1 \int_{\Omega^M \setminus \Omega^M_{2h}} \tilde{u}_{\varepsilon}^M(t, \bar{x} + \bar{\xi}, y) - \tilde{u}_{\varepsilon}^M(t, \bar{x}, y) \right| \right|^2 dyd\bar{x}dt
\]

\[
= \sum_{i \in I} \sum_{k \in \{0,1\}^{n-1}} \int_0^1 \int_{\varepsilon Y_i^k} \int_{\mathbb{R}^m} \left| \tilde{u}_{\varepsilon}^M(t, \bar{x} + \bar{\xi}, y) - \tilde{u}_{\varepsilon}^M(t, \bar{x}, y) \right|^2 dyd\bar{x}dt
\]

\[
\leq \sum_{i \in I} \sum_{k \in \{0,1\}^{n-1}} \int_0^1 \int_{\varepsilon Y_i^k} \int_{\mathbb{R}^m} \left| \tilde{u}_{\varepsilon}^M(t, \bar{x} + \bar{\xi}, y) - \tilde{u}_{\varepsilon}^M(t, \bar{x}, y) \right|^2 dyd\bar{x}dt
\]

\[
= \sum_{k \in \{0,1\}^{n-1}} \frac{1}{\varepsilon} \int_0^1 \int_{\Omega^M} \left| \tilde{u}_{\varepsilon}^M(t, \bar{x} + \bar{\xi} + k, y) - \tilde{u}_{\varepsilon}^M(t, \bar{x}, y) \right|^2 dyd\bar{x}dt.
\]
Let us fix $h \in \left(0, \min \left\{ \frac{1}{4}, \frac{\varepsilon}{2(n-1)} \right\} \right)$, where $C$ is the constant in the estimate (2.22). Let $\delta_1$ and $\varepsilon_1$ be such that for $|\xi| < \delta_1$ and $\varepsilon < \varepsilon_1$ we have

$$\left| \begin{bmatrix} \xi \\ \varepsilon \end{bmatrix} \varepsilon + k\varepsilon \right| < h.$$ 

Then we can apply the first part of the theorem to estimate

$$\sum_{k \in \{0,1\}^{n-1}} \frac{1}{\varepsilon} \int_0^T \int_{\Omega^M} \left| \hat{u}^M_\varepsilon \left( t, x + \varepsilon \left( \begin{bmatrix} \xi \\ \varepsilon \end{bmatrix} + k, 0 \right) \right) - \hat{u}^M_\varepsilon (t, x) \right|^2 \, dx dt$$

\leq \sum_{k \in \{0,1\}^{n-1}} C \left\| \hat{u}^+_\varepsilon \left( t, x + \varepsilon \left( \begin{bmatrix} \xi \\ \varepsilon \end{bmatrix} + k, 0 \right) \right) - \hat{u}^+_\varepsilon (t, x) \right\|^2_{L^2((0,T) \times \hat{S}_\varepsilon^+, \mathbb{R}^m)} + \sum_{k \in \{0,1\}^{n-1}} \sum_{i=1}^n \frac{C}{\varepsilon} \left\| \hat{U}^M_0 \left( \bar{x} + \varepsilon \left( \begin{bmatrix} \xi \\ \varepsilon \end{bmatrix} + k \right), \frac{x_n}{\varepsilon} \right) - \hat{U}^M_0 \left( \bar{x}, \frac{x_n}{\varepsilon} \right) \right\|^2_{L^2(\hat{\Omega}^M, \mathbb{R}^m)} + \sum_{k \in \{0,1\}^{n-1}} C \left\{ \frac{\varepsilon^2}{h^2} + h \right\}.

(5.17)

The properties of the initial concentration $U^M_0$ ensure that $\hat{U}^M_0$ is a function in $H^1$ on the rescaled domain obtained from $\hat{\Omega}^M$, and thus it satisfies the Kolmogorov criterion. This fact and the strong convergence of the traces of $u^\pm_\varepsilon$ on $S^\pm_\varepsilon$ from Proposition 2.2 imply that there exists $\delta^* > 0$, such that the sum over the differences of the boundary values and of the initial values in (5.17) is smaller than $\frac{\rho}{2}$ for

$$\left| \begin{bmatrix} \xi \\ \varepsilon \end{bmatrix} \varepsilon + k\varepsilon \right| < \delta^*.$$ 

This condition is fulfilled for all $\bar{\xi} = (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$, such that $|\xi_i| < \frac{\delta^*}{2\sqrt{n-1}}$ for $i = 1, \ldots, n-1$ and all $\varepsilon < \frac{\delta^*}{2\sqrt{n-1}}$.

Now, we set $\delta_2 = \min \{\delta_1, \frac{\delta^*}{2\sqrt{n-1}}\}$. Then, for all $\xi$ with $|\bar{\xi}| < \delta_2$ and all $\varepsilon < \varepsilon_2 = \min \{\varepsilon_0, \varepsilon_1, \frac{\delta^*}{2\sqrt{n-1}}, \sqrt{\frac{\delta^*}{3\sqrt{2n-1}}}, h\}$, we have that

$$\left\| L_\varepsilon \hat{u}^M_\varepsilon \left( t, \bar{x} + \bar{\xi}, y \right) - L_\varepsilon \hat{u}^M_\varepsilon \left( t, \bar{x}, y \right) \right\|_{L^2((0,T) \times \Sigma \times \mathbb{R}^m)} < \rho.$$ 

(5.18)
For \( \varepsilon \in (\varepsilon_2, \varepsilon_0) \), the estimate (5.16) holds for every \( \varepsilon \) if \( |\xi| < \delta(\varepsilon) \) due to the continuity in the mean of \( L^2 \)-functions. Since we consider sequences \( \varepsilon \) of the form \( \varepsilon_k = \frac{1}{k} \), \( k \in \mathbb{N} \), there are finitely many elements \( \varepsilon_k \) in the interval \( (\varepsilon_2, \varepsilon_0) \). Thus choosing

\[
\delta = \min \{ \delta_2, \delta(\varepsilon_k), \varepsilon_k \in (\varepsilon_2, \varepsilon_0) \},
\]

the property (5.16) is proved.

In addition, for \( L^2 \)-functions the following conditions are satisfied:

\[
(5.19) \quad \| \nabla_y L^2 u^M \|_{L^2((0, T) \times \Sigma \times Z, \mathbb{R}^m)} = \sqrt{\varepsilon} \| \nabla u^M \|_{L^2((0, T) \times \Omega^M, \mathbb{R}^m)} \leq C,
\]

\[
(5.20) \quad \| \partial_t L^2 u^M \|_{L^2((0, T) \times \Sigma \times Z, \mathbb{R}^m)} = \frac{1}{\sqrt{\varepsilon}} \| \partial_t u^M \|_{L^2((0, T) \times \Omega^M, \mathbb{R}^m)} \leq C.
\]

The conditions (5.16), (5.19), and (5.20) imply that the Kolmogorov criterion for \( L^2 \)-functions holds true in \( L^2((0, T) \times \Sigma \times Z, \mathbb{R}^m) \). This concludes the proof of our theorem. \( \square \)

**Lemma 5.1.** For \( u^M \in H^1(\Omega^M) \), the following trace estimate holds:

\[
(5.21) \quad \| u^M \|_{L^2(S^+ \cup S^-)} \leq C \left( \frac{1}{\sqrt{\varepsilon}} \| u^M \|_{L^2(\Omega^M)} + \sqrt{\varepsilon} \| \nabla u^M \|_{L^2(\Omega^M)} \right).
\]

**Proof.** The proof follows by a scaling argument and by the standard trace estimate for \( H^1 \)-functions. \( \square \)

**6. Derivation of the macroscopic model.** Using the convergence results proved in section 5, we are able to pass to the limit in the weak formulation of the microscopic problem.

**6.1. Derivation of the equations in the bulk.** First, we derive the macroscopic problem satisfied by the limit functions \( u^M \).

**Proof.** Let us consider test functions \( \varphi^\pm \in C_0^\infty ((0, T) \times \Omega, \mathbb{R}^m) \) with

\[
\text{supp } \varphi^\pm \subset (0, T) \times \Omega^\pm.
\]

Choose \( \varepsilon_0 \) such that

\[
(6.1) \quad \min \{ \text{dist } \Sigma, \text{supp } \varphi^+ \}, \text{dist } \Sigma, \text{supp } \varphi^- \} \geq \varepsilon_0.
\]

Then for every \( \varepsilon < \varepsilon_0 \) we have

\[
\text{supp } \varphi^\pm \cap \Omega^M = \emptyset.
\]

Testing now (2.16) with \( \varphi^+ \) and \( \varphi^- \), we obtain

\[
\int_0^T \int_{\Omega^M} \partial_t u^\pm_{j\varepsilon}(t, x) \varphi^\pm_j(t, x) dx dt + D_{\Omega^M}^\pm \int_0^T \int_{\Omega^M} \nabla u^\pm_{j\varepsilon}(t, x) \nabla \varphi^\pm_j(t, x) dx dt
\]

\[
= \int_0^T \int_{\Omega^M} f_j(x, u^\pm_{j\varepsilon}(t, x)) \varphi^\pm_j(t, x) dx dt.
\]

For \( \varepsilon \to 0 \) the terms on the left-hand side converge due to the weak compactness results from Proposition 2.1. The convergence of the right-hand side follows due to the following argument: By Proposition 2.1

\[
\chi_{\Omega^M_{j\varepsilon}} u_{j\varepsilon} \to u^\pm_{j0} \quad \text{strongly in } L^2((0, T), L^2(\Omega^\pm))
\]
and there exists a subsequence, again denoted by \( u^\pm_\varepsilon \), such that
\[
\chi_{\Omega^\pm} u_{j\varepsilon} \to u^\pm_{j0} \quad \text{a.e. in } (0, T) \times \Omega^\pm.
\]

Then the continuity of \( f \) implies
\[
f(\cdot, \chi_{\Omega^\pm} u_\varepsilon(\cdot, \cdot)) \to f(\cdot, u^\pm_{0}(\cdot, \cdot)) \quad \text{a.e. in } (0, T) \times \Omega^\pm.
\]

On the other hand, from the growth conditions (2.5) and the a priori estimates on \( u^\pm_\varepsilon \), we have
\[
\|f_j(\cdot, \chi_{\Omega^\pm} u_\varepsilon(\cdot, \cdot))\|_{L^2((0, T) \times \Omega^\pm)} = \int_0^T \int_{\Omega^\pm} |f_j(x, \chi_{\Omega^\pm} u_\varepsilon(t, x))|^2 dx \, dt
\]
\[
\leq \int_0^T \int_{\Omega^\pm} (1 + |\chi_{\Omega^\pm} u_\varepsilon(t, x)|^2) dx \, dt \leq C.
\]

Thus,
\[
f_j(\cdot, \chi_{\Omega^\pm} u_\varepsilon(\cdot, \cdot)) \to f_j(\cdot, u^\pm_{0}(\cdot, \cdot)) \quad \text{weakly in } L^2((0, T) \times \Omega^\pm).
\]

Now, taking the limit \( \varepsilon \to 0 \) in (6.2), we obtain
\[
\int_0^T \int_{\Omega^\pm} \partial_t u^\pm_{j0}(t, x) \varphi^\pm_j(t, x) dx \, dt + D^\pm_j \int_0^T \int_{\Omega^\pm} \nabla u^\pm_{j0}(t, x) \nabla \varphi^\pm_j(t, x) dx \, dt
\]
\[
= \int_0^T \int_{\Omega^\pm} f_j(x, u^\pm_{0}(t, x)) \varphi^\pm_j(t, x) dx \, dt,
\]

which is exactly the variational formulation for equations (2.23) and (2.24). The Dirichlet boundary condition can be deduced very easily, since \( u^\pm_\varepsilon|_{\partial_\varepsilon \Omega^\pm} = u^\pm_{j0}|_{\partial_\varepsilon \Omega^\pm} \). The Neumann boundary conditions are obtained by testing (2.16) with test functions \( \varphi^\pm \in C^\infty_0((0, T), C^\infty(\Omega, \mathbb{R}^m)) \) with \( \text{supp } \varphi^\pm \subset (0, T) \times \{\Omega^\pm \cup \partial_\varepsilon \Omega^\pm\} \) and by repeating the arguments from the first part of the proof.

It remains to derive the initial conditions for \( u^\pm_0 \). Thus, let us consider \( \varphi^\pm \in C^\infty_0(\Omega^\pm, \mathbb{R}^m) \) and again choose \( \varepsilon_0 > 0 \) as in (6.1). Then let \( \xi \in C^\infty([0, T]) \), such that \( \xi(T) = 0 \). Then for every \( \varepsilon < \varepsilon_0 \) we have
\[
\int_0^T \int_{\Omega^\pm} \partial_t u^\pm_{j0}(t, x) \varphi^\pm_j(x, t) \xi(t) dx \, dt = - \int_{\Omega^\pm} U_{j0}(x) \varphi^\pm_j(x) \xi(0) dx
\]
\[
- \int_0^T \int_{\Omega^\pm} u^\pm_{j0}(t, x) \varphi^\pm_j(x) \partial_t \xi(t) dx \, dt.
\]

Proposition 2.1 and the choice of our test function allow us to pass to the limit for \( \varepsilon \to 0 \) and to obtain
\[
\int_0^T \int_{\Omega^\pm} \partial_t u^\pm_{j0}(x) \varphi^\pm_j(x, t) \xi(t) dx \, dt = - \int_{\Omega^\pm} U_{j0}(x) \varphi^\pm_j(x) \xi(0) dx
\]
\[
- \int_0^T \int_{\Omega^\pm} u^\pm_{j0}(t, x) \varphi^\pm_j(x) \partial_t \xi(t) dx \, dt,
\]

which is equivalent to the initial conditions (2.27). \(\square\)
6.2. Derivation of the local equations in the layer. We now derive the local problem for the limit function $u_0^M$ which enters the transmission conditions.

Proof. We start from the weak formulation (2.16) and use as test function

$$\varphi^\varepsilon(t,x) = \begin{cases} 0, & (t,x) \in (0,T) \times (\Omega^+_\varepsilon \cup \Omega^-_\varepsilon), \\ \varphi(t,\bar{x},\bar{x}/\varepsilon), & (t,x) \in (0,T) \times \Omega^M_\varepsilon, \end{cases}$$

where $\varphi \in \{C_0^\infty((0,T) \times \Sigma, C_0^\infty(Y, C_0^\infty([-1,1]))\}^m$. Thus, we obtain

$$\frac{1}{\varepsilon} \int_0^T \int_{\Omega^M_\varepsilon} \partial_t u_j^M \varphi_j \left( t, \bar{x}, \frac{x}{\varepsilon} \right) dx dt$$

$$+ \frac{1}{\varepsilon} \int_0^T \int_{\Omega^M_\varepsilon} D_j^M \left( \frac{x}{\varepsilon} \right) \varepsilon \nabla u_j^M \left( \varepsilon \nabla \varphi_j \left( t, \bar{x}, \frac{x}{\varepsilon} \right) + \nabla_y \varphi_j \left( t, \bar{x}, \frac{x}{\varepsilon} \right) \right) dx dt$$

$$= \frac{1}{\varepsilon} \int_0^T \int_{\Omega^M_\varepsilon} g_j \left( \frac{x}{\varepsilon}, u_j^M \right) \varphi_j \left( t, \bar{x}, \frac{x}{\varepsilon} \right) dx dt. \tag{6.3}$$

Using the weak two-scale convergence properties from Proposition 2.1, we can pass to the limit for $\varepsilon \to 0$ on the left-hand side of the equation above. In order to pass to the limit in the nonlinear term on the right-hand side, we have to use the strong two-scale convergence given in Theorem 2.3 to show that

$$g_j \left( \frac{x}{\varepsilon}, u_j^M \right) \xrightarrow{\text{two-scale}} g_j(y, u_0^M) \quad \text{weakly in the two-scale sense.}$$

Thus, let us consider

$$L_\varepsilon \left[ g_j \left( \frac{x}{\varepsilon}, u_j^M \right) \right](t,\bar{x},y) = g_j \left( \frac{c_x(\bar{x}) + \varepsilon \bar{y}, \varepsilon y_n}{\varepsilon}, u_j^M(\varepsilon, c_x(\bar{x}) + \varepsilon \bar{y}, \varepsilon y_n) \right)$$

$$= g_j(y, L_\varepsilon u_j^M(t,\bar{x},y)).$$

Since $g$ is continuous and $L_\varepsilon u_j^M$ converges strongly to $u_0^M$ in $L^2((0,T) \times \Sigma \times Z, \mathbb{R}^m)$, we have that

$$g_j(\cdot, L_\varepsilon u_j^M) \longrightarrow g_j(\cdot, u_0^M) \quad \text{a.e. in } (0,T) \times \Sigma \times Z.$$

From the growth conditions (2.6) and the a priori estimates on $u_j^M$ we obtain

$$\|g_j(\cdot, L_\varepsilon u_j^M)\|_{L^2((0,T) \times \Sigma \times Z)} = \int_0^T \int_{\Omega^M_\varepsilon} \int_Z |g_j(y, u_j^M(\varepsilon, c_x(\bar{x}) + \varepsilon \bar{y}, \varepsilon y_n))|^2 dy d\bar{x} dt$$

$$\leq C \int_0^T \int_{\Omega^M_\varepsilon} \int_Z 1 + |u_j^M(\varepsilon, c_x(\bar{x}) + \varepsilon \bar{y}, \varepsilon y_n))|^2 dy d\bar{x} dt$$

$$\leq C_1 + \frac{C_2}{\varepsilon} \|u_j^M\|_{L^2((0,T) \times \Omega^M_\varepsilon, \mathbb{R}^m)}^2 \leq C. \tag{6.4}$$

Thus,

$$g_j(\cdot, L_\varepsilon u_j^M) \longrightarrow g_j(\cdot, u_0^M) \quad \text{weakly in } L^2((0,T) \times \Sigma \times Z).$$

This now implies that

$$L_\varepsilon \left[ g_j \left( \frac{x}{\varepsilon}, u_j^M \right) \right] \rightarrow g_j(\cdot, u_0^M) \quad \text{weakly in } L^2((0,T) \times \Sigma \times Z),$$

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and by Theorem 4.7 it follows that
\[ g_j \left( \frac{\cdot}{\varepsilon}, u^M_\varepsilon \right) \xrightarrow{L^2} g_j(y, u^M_0) \]
weakly in the two-scale sense.

Thus, we can pass to the limit in (6.3) and obtain
\[
\int_0^T \int_\Sigma \int_Z \partial_t u^M_{j0}(t, \bar{x}, y) \varphi_j(t, \bar{x}, y) \, dy \, d\bar{x} \, dt \\
+ \int_0^T \int_\Sigma \int_Z D^M_j(y) \nabla_y u^M_{j0}(t, \bar{x}, y) \nabla_y \varphi_j(t, \bar{x}, y) \, dy \, d\bar{x} \, dt \\
= \int_0^T \int_\Sigma \int_Z g_j(y, u^M_0(t, \bar{x}, y)) \varphi_j(t, \bar{x}, y) \, dy \, d\bar{x} \, dt.
\]

This is just the weak formulation of (2.30). The boundary conditions have already been proved in Proposition 2.1; see (2.19). It remains to prove the initial condition for \( u^M_0 \). We will proceed as in section 6.1.

Let \( \varphi \in \{ C^\infty_0(\Sigma, C^\infty_{per}(Y, C^\infty_0([-1,1])) \}^m \) and \( \xi \in C^\infty([0, T]) \) such that \( \xi(T) = 0 \). Then the following relation holds:
\[
\frac{1}{\varepsilon} \int_{\Omega^M_\varepsilon} \int_0^T \partial_t u^M_{\varepsilon}(t, x) \varphi_j \left( \frac{x}{\varepsilon}, \frac{\bar{x}}{\varepsilon} \right) \xi(t) \, dx \, dt = -\frac{1}{\varepsilon} \int_{\Omega^M_\varepsilon} U^M_{j0} \left( \frac{\bar{x}}{\varepsilon}, \frac{x}{\varepsilon} \right) \varphi_j \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon} \right) \xi(0) \, dx \\
- \frac{1}{\varepsilon} \int_0^T \int_{\Omega^M_\varepsilon} u^M_{\varepsilon}(t, x) \varphi_j \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon} \right) \partial_t \xi(t) \, dx \, dt.
\]

Using the convergence results from Proposition 2.1 together with Lemma 4.3, we can pass to the limit for \( \varepsilon \to 0 \) and obtain
\[
\int_0^T \int_\Sigma \int_Z \partial_t u^M_{j0}(t, \bar{x}, y) \varphi(\bar{x}, y) \xi(t) \, dy \, d\bar{x} \, dt = -\int_\Sigma \int_Z U^M_{j0}(\bar{x}, y) \varphi(\bar{x}, y) \xi(0) \, dy \, d\bar{x} \\
- \int_0^T \int_\Sigma \int_Z u^M_{j0}(t, \bar{x}, y) \varphi_j(\bar{x}, y) \partial_t \xi(t) \, dy \, d\bar{x} \, dt,
\]
which immediately implies the initial conditions (2.33) for the local problem.

\[ \Box \]

6.3. Derivation of the transmission conditions across \( \Sigma \). Measurements and experiments indicate jumps of the concentrations and their normal derivatives across the interface \( \Sigma \). We derive formulas for these jumps which involve the solutions of the local cell problems. Thus, the jumps become computable.

**Definition 6.1.** For \( u \in L^2((0, T) \times \Omega, \mathbb{R}^m) \) with \( u^\pm \in L^2((0, T), H^1(\Omega^\pm, \mathbb{R}^m)) \), the jump of \( u \) across \( \Sigma \) is defined as
\[
[u]_{\Sigma}(t, \bar{x}) = u^+(t, \bar{x}, 0) - u^-(t, \bar{x}, 0), \quad (t, \bar{x}) \in (0, T) \times \Sigma.
\]

To find a relation for the jump in the concentrations, we use the boundary layer function constructed below.

Consider the Hilbert space
\[ V = \{ \varphi \in H^1(Z, \mathbb{R}^m), \varphi \text{ periodic in } Y, \varphi \equiv \text{ const on } S^+ \cup S^- \}. \]

Find \( \eta \in V \) such that
\[
\frac{1}{|Z|} \int_Z \eta(y) \, dy = 0
\]
Fig. 6.1. The boundary layer \( \eta_j(y) \) for \( D_j(y) \equiv 1 \).

and, for all \( \varphi \in V \),

\[
\int_Z D_j^M(y) \nabla \eta_j(y) \nabla \varphi_j(y) \, dy = \int_{S^+} \varphi_j(y) \, ds - \int_{S^-} \varphi_j(y) \, ds.
\]

Since the right-hand side of (6.7) defines a linear, continuous functional on \( V \), the
Lax–Milgram lemma implies the existence of a unique solution \( \eta \in V \) to problem
(6.7) with (6.6). The strong formulation for the problem for \( \eta \) is given as follows:
Find \( \eta \in V \) with \( \frac{1}{|Z|} \int_Z \eta(y) \, dy = 0 \) such that

\[
\nabla_y (D_j^M(y) \nabla_y \eta_j(y)) = 0, \quad y \in Z,
\]

\[
\int_{S^+} D_j^M(y) \partial_n \eta_j(y) \, dy = 1,
\]

\[
\int_{S^-} D_j^M(y) \partial_n \eta_j(y) \, dy = -1.
\]

In the case \( D_j^M(y) \equiv 1 \), the function \( \eta \) is given by (see Figure 6.1)

\[
\eta_j(y) = y_n, \quad j = 1, \ldots, m.
\]

We denote

\[
\eta^\pm \in \mathbb{R}^m \text{ the constant values of } \eta \text{ on } S^\pm.
\]

The derivation of the transmission condition is now possible by using a test func-
tion of boundary layer type as shown below.

Proof. Consider the function

\[
\eta_\varepsilon(x) = \begin{cases} 
\eta^+, & x \in \Omega^+_\varepsilon, \\
\eta_\varepsilon(x), & x \in \Omega^M_\varepsilon, \\
\eta^-, & x \in \Omega^-_\varepsilon,
\end{cases}
\]

and let \( \xi(t, \bar{x}) \) be a smooth function, \( \xi \in C_0^\infty((0, T) \times \Sigma, \mathbb{R}^m) \). To get the jump
condition for \( u_{j0} \) across \( \Sigma \), we evaluate the integral

\[
I_\varepsilon = \int_0^T \int_{\Omega^M_\varepsilon} \varepsilon D_j^M \left( \frac{x}{\varepsilon} \right) \nabla \eta_\varepsilon(x) \nabla (u^M_{j0}(t, x, \xi_j(t, \bar{x}))) \, dx \, dt
\]

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in two different ways and then pass to the limit for \( \varepsilon \to 0 \). First, integrating by parts and taking into account (6.8) and the continuity of \( u \) across \( S^+_\varepsilon \) and \( S^-_\varepsilon \), we obtain

\[
I_\varepsilon = \int_0^T \int_{S^+_\varepsilon} \varepsilon \mathcal{D}^M_j \left( \frac{x}{\varepsilon} \right) \frac{1}{\varepsilon} \nabla y_j \left( \frac{x}{\varepsilon} \right) \cdot \vec{v}(x) \ u^+_j(t,x) \xi_j(t,x) dx dt
\]

\[
+ \int_0^T \int_{S^-_\varepsilon} \varepsilon \mathcal{D}^M_j \left( \frac{x}{\varepsilon} \right) \frac{1}{\varepsilon} \nabla y_j \left( \frac{x}{\varepsilon} \right) \cdot \vec{v}(x) \ u^-_j(t,x) \xi_j(t,x) dx dt.
\]

Now, Proposition 2.2 allows us to pass to the limit for \( \varepsilon \to 0 \) and to obtain

\[
\lim_{\varepsilon \to 0} I_\varepsilon = \int_0^T \int_{\Sigma} u^+_j(t,\bar{x},0) \xi_j(t,\bar{x}) \mathcal{D}^M_j(\bar{y},1) \partial_n \eta_j(\bar{y},1) d\bar{y} d\bar{x} dt
\]

\[
- \int_0^T \int_{\Sigma} u^-_j(t,\bar{x},0) \xi_j(t,\bar{x}) \mathcal{D}^M_j(\bar{y},-1) \partial_n \eta_j(\bar{y},-1) d\bar{y} d\bar{x} dt
\]

\[
= \int_0^T \int_{\Sigma} (u^+_j(t,\bar{x},0) - u^-_j(t,\bar{x},0)) \xi_j(t,\bar{x}) d\bar{x} dt,
\]

where the last equality follows from the boundary conditions (6.9) and (6.10) for \( \eta \).

Second, by differentiation in the second gradient, we have

\[
I_\varepsilon = \int_0^T \int_{\Omega^\varepsilon} \varepsilon \mathcal{D}^M_j \left( \frac{x}{\varepsilon} \right) \nabla \eta_j(x) \nabla \xi_j(t,\bar{x}) u^M_j(t,x) dx dt
\]

\[
+ \int_0^T \int_{\Omega^\varepsilon} \varepsilon \mathcal{D}^M_j \left( \frac{x}{\varepsilon} \right) \nabla (\eta_j(x) \xi_j(t,\bar{x})) \nabla u^M_j(t,x) dx dt
\]

\[
- \int_0^T \int_{\Omega^\varepsilon} \varepsilon \mathcal{D}^M_j \left( \frac{x}{\varepsilon} \right) \eta_j(x) \nabla \xi_j(t,\bar{x}) \nabla u^M_j(t,x) dx dt
\]

\[
= I^1_\varepsilon + I^2_\varepsilon + I^3_\varepsilon.
\]

Since \( u^M_j \) and \( \varepsilon \nabla u^M_j \) converge weakly in the two-scale sense (see Proposition 2.1), the integrals \( I^1_\varepsilon \) and \( I^3_\varepsilon \) converge to zero for \( \varepsilon \to 0 \). To calculate \( \lim_{\varepsilon \to 0} I^2_\varepsilon \), we start from the weak formulation (2.16) where we insert as test function

\[
(6.14) \quad \varphi(t,x) = \eta_j(x) \xi_j(t,\bar{x}) \psi(x_n)
\]

with \( \psi \in C^\infty_0((-H,H),\mathbb{R}) \), such that \( \psi(x_n) \equiv 1 \) for \( |x_n| < \frac{H}{2} \). Thus, we have

\[
\lim_{\varepsilon \to 0} I^2_\varepsilon = \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega^\varepsilon} \left( f_j(x,u^\varepsilon_j) - \partial_t u^\varepsilon_j \right) \partial_t \eta^+_j \xi_j(t,\bar{x}) \psi(x_n) dx dt
\]

\[
+ \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega^\varepsilon} \left( f_j(x,u^-_j) - \partial_t u^-_j \right) \partial_t \eta^-_j \xi_j(t,\bar{x}) \psi(x_n) dx dt
\]

\[
+ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^T \int_{\Omega^\varepsilon} \left( g_j(x,u^\varepsilon_j) - \partial_t u^M_j \right) \eta_j \left( \frac{x}{\varepsilon} \right) \xi_j(t,\bar{x}) dx dt
\]

\[
- \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega^\varepsilon} \partial_t \eta^+_j \nabla u^\varepsilon_j \nabla (\xi_j(t,\bar{x}) \psi(x_n)) dx dt
\]

\[
- \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega^\varepsilon} \partial_t \eta^-_j \nabla u^-_j \nabla (\xi_j(t,\bar{x}) \psi(x_n)) dx dt.
\]
Using the convergence properties from Proposition 2.1, Theorem 2.3, and the macroscopic problem for the limit functions $u_0^+$ and $u_0^-$, we obtain

$$
\lim_{\varepsilon \to 0} I^2_\varepsilon = \int_0^T \int_{\Omega^+} \int_Z (g_j(y, u_0^M(t, \bar{x}, y)) - \partial_t u_0^M(t, \bar{x}, y)) \eta_j(y) dy \xi_j(t, \bar{x}) d\bar{x} dt + \int_0^T \int_{\Sigma} (D^+\eta_j^+ \partial_n u_0^+(t, \bar{x}, 0) - D^-\eta_j^- \partial_n u_0^-(t, \bar{x}, 0)) \xi_j(t, \bar{x}) d\bar{x} dt.
$$

Thus, the first transmission condition is proved. It remains to derive a transmission condition for the fluxes. We start again from the weak formulation (2.16) and use as test function

$$
\varphi(t, x) = \xi(t, \bar{x}) \psi(x_n),
$$

with $\xi$ and $\psi$ as in (6.14). For $\varepsilon \to 0$ we obtain

$$
\int_0^T \int_{\Omega^+} \partial_x u_0^+ \xi_j \psi_j dx dt + \int_0^T \int_{\Omega^+} D^+ \nabla u_0^+ \nabla (\xi_j \psi_j) dx dt - \int_0^T \int_{\Omega^+} f_j(x, u_0^+) \xi_j \psi_j dx dt + \int_0^T \int_{\Omega^-} \partial_x u_0^- \xi_j \psi_j dx dt + \int_0^T \int_{\Omega^-} D^- \nabla u_0^- \nabla (\xi_j \psi_j) dx dt - \int_0^T \int_{\Omega^-} f_j(x, u_0^-) \xi_j \psi_j dx dt + \int_0^T \int_{\Sigma} \int_Z (\partial_x u_0^+(t, \bar{x}, y) - g_j(y, u_0^M(t, \bar{x}, y))) dy \xi_j(t, \bar{x}) d\bar{x} dt = 0.
$$

Using again the macroscopic equations for $u_0^+$ and $u_0^-$, we get the following transmission condition:

$$
\int_0^T \int_{\Sigma} (D^+ \partial_n u_0^+(t, \bar{x}, 0) - D^- \partial_n u_0^-(t, \bar{x}, 0)) \xi_j(t, \bar{x}) = \int_0^T \int_{\Sigma} \int_Z (\partial_x u_0^M(t, \bar{x}, y) - g_j(y, u_0^M(t, \bar{x}, y))) dy \xi_j(t, \bar{x}) d\bar{x} dt. \quad \Box
$$

### 7. Uniqueness for the macroscopic model

In this section, we want to show uniqueness of the solutions $(u_0^+, u_0^-, u_0^M)$ to the effective model presented in Theorem 2.4. To this end, let us first formulate a corollary which gives an equivalent formulation of the transmission conditions (2.28), (2.29).

**Corollary 7.1.** The transmission conditions (2.28), (2.29) are equivalent to

1. $$
D^+ \partial_n u_0^+(t, \bar{x}, 0) = \int_{S^+} D_j^M(y) \partial_n u_0^M(t, \bar{x}, y) dy,
$$
2. $$
D^- \partial_n u_0^-(t, \bar{x}, 0) = \int_{S^-} D_j^M(y) \partial_n u_0^M(t, \bar{x}, y) dy.
$$

These equivalent transmission conditions hold in a distributional sense with respect to $t$ and $\bar{x}$.

**Remark 2.** The physical interpretation of (7.1) and (7.2) is obvious: it states that the macroscopic flux is given by the microscopic flux averaged over the corresponding part of the cell surface.

**Proof.** Using the properties (6.8), (6.9), and (6.10) of the boundary layer $\eta$ and the boundary conditions

$$
u_0^M(t, \bar{x}, y) = u_0^\pm(t, \bar{x}, 0) \text{ on } (0, T) \times \Sigma \times S^\pm,
$$

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we obtain
\[
[u_{j0}]_{\Sigma}(t, \bar{x}) = u_{j0}^+(t, \bar{x}, 0) - u_{j0}^-(t, \bar{x}, 0)
\]
\[
= \int_{S^+} u_{j0}^M(t, \bar{x}, y)dy - \int_{S^-} u_{j0}^M(t, \bar{x}, y)dy
\]
\[
= \int_{Z} D_j^M(y)\nabla \eta_j(y)\nabla u_{j0}^M(t, \bar{x}, y)dy.
\]

Equation (2.30) for \(u_0^M\) and the fact that \(\eta\) is constant on \(S^+\) and \(S^-\) with constants defined in (6.11) yield

\[(7.3)\quad [u_{j0}]_{\Sigma}(t, \bar{x}) = \int_{Z} (g_j(y, u_0^M(t, \bar{x}, y)) - \partial_t u_{j0}^M(t, \bar{x}, y))\eta_j(y) dy
\]
\[
+ \int_{S^+} D_j^M(y)\partial_n u_{j0}^M(t, \bar{x}, y)\eta_j^+dy - \int_{S^-} D_j^M(y)\partial_n u_{j0}^M(t, \bar{x}, y)\eta_j^-dy.
\]

From (2.28) and (7.3), we then obtain

\[(7.4)\quad D_j^+\partial_n u_{j0}^+(t, \bar{x}, 0)\eta_j^+ - D_j^-\partial_n u_{j0}^-(t, \bar{x}, 0)\eta_j^-
\]
\[
= \int_{S^+} D_j^M(y)\partial_n u_{j0}^M(t, \bar{x}, y)\eta_j^+dy - \int_{S^-} D_j^M(y)\partial_n u_{j0}^M(t, \bar{x}, y)\eta_j^-dy.
\]

On the other hand, using (2.30) for \(u_0^M\), the transmission condition (2.29) transforms to

\[(7.5)\quad D_j^+\partial_n u_{j0}^+(t, \bar{x}, 0) - D_j^-\partial_n u_{j0}^-(t, \bar{x}, 0)
\]
\[
= \int_{S^+} D_j^M(y)\partial_n u_{j0}^M(t, \bar{x}, y)dy - \int_{S^-} D_j^M(y)\partial_n u_{j0}^M(t, \bar{x}, y)dy.
\]

Relations (7.4) and (7.5) are equivalent with the transmission conditions (7.1) and (7.2) since \(\eta_j^+ - \eta_j^- \neq 0\). This follows from the relations

\[\eta_j^+ > 0 \quad \text{and} \quad \eta_j^- < 0 \quad \text{for} \quad j = 1, \ldots, n,
\]

which hold by the maximum principle for elliptic equations. \(\Box\)

In the following, we prove uniqueness for the macroscopic system by using the equivalent transmission conditions (7.1) and (7.2).

Proof. Assume that \((u_i^+, u_i^-, u_i^M), i = 1, 2,\) are solutions of the macroscopic system with the same data. Let \(\delta u^+, \delta u^-, \delta u^M\) denote the differences. Now, we consider a test function \(\varphi \in V,\) where

\[V = \{(C_0^\infty((0, T), C^\infty(\overline{\Omega}^+)))^m \cap (C_0^\infty((0, T), C^\infty(\overline{\Omega}^-)))^m, \varphi = 0 \text{ on } \partial_D \Omega^+ \cup \partial_D \Omega^-\}.
\]

We multiply (2.23) and (2.24) by \(\varphi\) and integrate, obtaining

\[(7.6)\quad \int_{\Omega^+} \delta_t u_{j0}^+ \varphi_j^+ dx + \int_{\Omega^-} \delta_t u_{j0}^- \varphi_j^- dx + D_j^+ \int_{\Omega^+} \nabla u_{j0}^+ \nabla \varphi_j^+ dx
\]
\[
+ D_j^- \int_{\Omega^-} \nabla u_{j0}^- \nabla \varphi_j^- dx + \int_{\Sigma} D_j^+ \partial_n u_{j0}^+ \varphi_j^+ ds(x) - \int_{\Sigma} D_j^- \partial_n u_{j0}^- \varphi_j^- ds(x)
\]
\[
= \int_{\Omega^+} f_j(x, u_0^+) \varphi_j^+ dx + \int_{\Omega^-} f_j(x, u_0^-) \varphi_j^- dx.
\]
From (7.6) and the transmission conditions (7.1) and (7.2), we obtain the following equation for the differences $\delta u^{+}, \delta u^{-}$:

\[
\int_{\Omega^{+}} \partial_{i}(\delta u^{+}_{j})\varphi^{+}_{j} dx + \int_{\Omega^{-}} \partial_{i}(\delta u^{-}_{j})\varphi^{-}_{j} dx + D^{+}_{j} \int_{\Omega^{+}} \nabla(\delta u^{+}_{j})\nabla\varphi^{+}_{j} dx \\
+ \int_{\Omega^{-}} \nabla(\delta u^{-}_{j})\nabla\varphi^{-}_{j} dx + \int_{\Sigma} \int_{S^{+}} D^{M}(y)\partial_{n}(\delta u^{M}_{j}) dy \varphi^{+}_{j} d\Sigma(x) \\
\int_{\Sigma} \int_{S^{-}} D^{M}(y)\partial_{n}(\delta u^{M}_{j}) dy \varphi^{-}_{j} d\Sigma(x) = \int_{\Omega^{+}} \delta f^{+}_{j}\varphi^{+}_{j} dx + \int_{\Omega^{-}} \delta f^{-}_{j}\varphi^{-}_{j} dx,
\]

(7.7)

where we denoted

\[
\delta f^{+}_{j} = f_{j}(x, u^{+}_{1}) - f_{j}(x, u^{+}_{2}), \quad \delta f^{-}_{j} = f_{j}(x, u^{-}_{1}) - f_{j}(x, u^{-}_{2}).
\]

Now, we insert $\varphi^{+} = \delta u^{+}$ and $\varphi^{-} = \delta u^{-}$ as test functions. Using the boundary conditions (2.31), we get for the terms on $\Sigma$

\[
\int_{\Sigma} \int_{S^{+}} D^{M}(y)\partial_{n}(\delta u^{M}_{j}) dy \delta u^{+}_{j} d\Sigma - \int_{\Sigma} \int_{S^{-}} D^{M}(y)\partial_{n}(\delta u^{M}_{j}) dy \delta u^{-}_{j} d\Sigma \\
= \int_{\Sigma} \int_{Z} D^{M}(y)\nabla(\delta u^{M}_{j}) \nabla(\delta u^{M}_{j}) dy d\Sigma + \int_{\Sigma} \int_{Z} \partial_{i}(\delta u^{M}_{j}) \delta u^{M}_{j} dy d\Sigma \\
- \int_{\Sigma} \int_{Z} \delta g_{j} \delta u^{M}_{j} dy d\Sigma.
\]

(7.8)

Here, we used the notation $\delta g_{j} = g_{j}(y, u^{M}_{1}) - g_{j}(y, u^{M}_{2})$. Inserting (7.8) in (7.7), summing up for $j = 1, \ldots, m$, and integrating with respect to $t$, we obtain

\[
\frac{1}{2}||\delta u^{+}(t)||^{2}_{L^{2}(\Omega^{+})} + \frac{1}{2}||\delta u^{-}(t)||^{2}_{L^{2}(\Omega^{-})} + \frac{1}{2}||\delta u^{M}(t)||^{2}_{L^{2}(\Sigma \times Z)} \\
+ D^{+}_{j} \int_{0}^{t} \int_{\Omega^{+}} |\nabla(\delta u^{+})|^{2} dx dt + D^{-}_{j} \int_{0}^{t} \int_{\Omega^{-}} |\nabla(\delta u^{-})|^{2} dx dt \\
+ \int_{0}^{t} \int_{\Sigma} \int_{Z} D^{M}(y)|\nabla(\delta u^{M})|^{2} dy d\Sigma dt = \int_{0}^{t} \int_{\Omega^{-}} \delta g_{j} \delta u^{M} dy d\Sigma dt \\
+ \int_{0}^{t} \int_{\Omega^{+}} \delta f^{+}\delta u^{+} dx dt + \int_{0}^{t} \int_{\Omega^{-}} \delta f^{-}\delta u^{-} dx dt.
\]

(7.9)

Using the Lipschitz continuity of the reaction terms, the right-hand side in (7.9) can be estimated by

\[
C \left( ||\delta u^{M}||^{2}_{L^{2}(\Omega^{+})} + ||\delta u^{+}||^{2}_{L^{2}(\Omega^{+})} + ||\delta u^{-}||^{2}_{L^{2}(\Omega^{-})} \right).
\]

Then Gronwall’s inequality yields

\[
\delta u^{+} = \delta u^{-} = \delta u^{M} = 0
\]

and the theorem is proved. □

**Corollary 7.2.** The entire sequence $(u^{+}_{e}, u^{-}_{e}, u^{M}_{e})$ converges to the limit $(u^{+}_{0}, u^{-}_{0}, u^{M}_{0})$, solving the macroscopic system.

**Remark 3.** An important aim of our homogenization procedure is to reduce the computational complexity. The algorithms for solving the derived transmission problem numerically will be considered in a forthcoming paper.
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