Abelian varieties and the inverse Galois problem

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The inverse Galois problem

Let G be a finite group. Does there exist a Galois extension $K/\mathbb{Q}$ such that $\mathrm{Gal}(K/\mathbb{Q}) \cong G$?

For example, let G be S_n, the symmetric group of n letters. Then G is a Galois group over $\mathbb{Q}$. Moreover, for all positive integer n we can realize G as the Galois group of the spliting field $x^n - x - 1$.

Galois representations may answer the inverse Galois problem for finite linear groups.

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ and let $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Let A be a principally polarized abelian variety over $\mathbb{Q}$ of dimension d.

Let ℓ be a prime and $A[\ell]$ the ℓ-torsion subgroup:

$$A[\ell] := \{P \in A(\overline{\mathbb{Q}}) \mid [\ell]P = 0\} \cong (\mathbb{Z}/\ell\mathbb{Z})^{2d}.$$

$A[\ell]$ is a $2d$-dimensional $\mathbb{F}_\ell$-vector space, as well as a $G_{\mathbb{Q}}$-module.

The polarization induces a symplectic pairing, the mod ℓ **Weil pairing** on $A[\ell]$, which is a bilinear, alternating, non-degenerate pairing:

$$\langle\ ,\ \rangle\ :\ A[\ell] \times A[\ell] \to \mu_\ell$$

that is Galois invariant: $\forall \sigma \in G_{\mathbb{Q}}\,,\ \forall v, w \in A[\ell]$

$$\langle \sigma v, \sigma w \rangle = \chi(\sigma)\langle v, w \rangle,$$

where $\chi : G_{\mathbb{Q}} \to \mathbb{F}_\ell^\times$ is the mod ℓ cyclotomic character.

$(A[\ell], \langle\ ,\ \rangle)$ is a symplectic $\mathbb{F}_\ell$-vector space of dimension $2d$. This gives a representation

$$\overline{\rho}_{A,\ell} : G_{\mathbb{Q}} \to \mathsf{GSp}(A[\ell], \langle\ ,\ \rangle) \cong \mathsf{GSp}_{2d}(\mathbb{F}_\ell).$$

Theorem (Serre)

Let $A/\mathbb{Q}$ be a principally polarized abelian variety of dimension d. Assume that $d = 2$, 6 or d is odd and, furthermore, assume that $\mathrm{End}_{\overline{\mathbb{Q}}}(A) = \mathbb{Z}$. Then there exists a bound B_A such that for all primes $\ell > B_A$ the representation $\overline{\rho}_{A,\ell}$ is surjective.

The conclusion of the theorem is known to be false for general d (counterexample by Mumford for $d = 4$).

Open question

Given d as in the theorem, is there a **uniform bound** B_d depending only on d, such that for all principally polarized abelian varieties A over $\mathbb{Q}$ of dimension d with $\mathrm{End}_{\overline{\mathbb{Q}}}(A) = \mathbb{Z}$, and all $\ell > B_d$, the representation $\overline{\rho}_{A,\ell}$ is surjective?

For elliptic curves an affirmative answer is expected, and this is known as Serre's Uniformity Question.

Much easier for semistable elliptic curves:

Theorem (Serre)

Let $E/\mathbb{Q}$ be a semistable elliptic curve, and $\ell \geq 11$ be a prime. Then $\overline{\rho}_{E,\ell}$ is surjective.

Inverse Galois problem: genus 1 & 2

The Galois representation attached to the ℓ-torsion of the **elliptic curve** $y^2 + y = x^3 - x$ is surjective for all prime ℓ. This gives a realization $\mathrm{GL}_2(\mathbb{F}_\ell)$ as Galois group for all prime ℓ.

Let C be the **genus** 2 **hyperelliptic curve** given by $y^2 = x^5 - x + 1$ and let J denotes its Jacobian. Dieulefait proved that $\overline{\rho}_{J,\ell}$ is surjective for all odd prime ℓ. This gives a realization $\mathrm{GSp}_4(\mathbb{F}_\ell)$ as Galois group for all odd prime ℓ.

Question

What about higher genus?

Theorem (A., Lemos and Siksek)

Let A be a semistable principally polarized abelian variety of dimension $d \geq 1$ over $\mathbb{Q}$ and let $\ell \geq \max(5, d+2)$ be prime.
Suppose the image of $\overline{\rho}_{A,\ell} : G_{\mathbb{Q}} \to \mathrm{GSp}_{2d}(\mathbb{F}_\ell)$ contains a transvection.
Then $\overline{\rho}_{A,\ell}$ is either reducible or surjective.

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Transvection

Definition

Let $(V, \langle\ ,\ \rangle)$ be a finite-dimensional symplectic vector space over $\mathbb{F}_\ell$. A **transvection** *is an element $T \in \mathrm{GSp}(V, \langle\ ,\ \rangle)$ which fixes a hyperplane $H \subset V$.*

Therefore, a transvection is a unipotent element $\sigma \in \mathrm{GSp}(V, \langle\ ,\ \rangle)$ such that $\sigma - I$ has rank 1.

When does $\overline{\rho}_{A,\ell}(G_{\mathbb{Q}})$ contain a transvection?

Let $q \neq \ell$ be a prime and suppose that the following two conditions are satisfied:

- the special fibre of the Néron model for A at q has **toric dimension** 1;
- $\ell \nmid \#\Phi_q$, where Φ_q is the group of connected components of the special fibre of the Néron model at q.

Then the image of $\overline{\rho}_{A,\ell}$ contains a **transvection** (Grothendieck, Hall).

When does $\overline{\rho}_{A,\ell}(G_{\mathbb{Q}})$ contain a transvection?

Let $C/\mathbb{Q}$ be a hyperelliptic curve of genus d:

$$C : y^2 = f(x)$$

where $f \in \mathbb{Z}[x]$ is a squarefree polynomial.

Let p be an odd prime not dividing the leading coefficient of f such that f modulo p has one root in $\overline{\mathbb{F}}_p$ having multiplicity precisely 2, with all other roots simple.
Then the Néron model of the Jacobian at p has toric dimension 1 (Grothendieck, Hall).

Ingredients of the proof of the main theorem

In the proof of this theorem we rely on:

- the classification due to Arias-de-Reyna, Dieulefait and Wiese of subgroups of $\mathrm{GSp}_{2d}(\mathbb{F}_\ell)$ containing a transvection;
- results of Raynaud on the image of the inertia subgroup.

Classification of subgroups of $\mathsf{GSp}_{2d}(\mathbb{F}_\ell)$ with a transvection

Theorem (Arias-de-Reyna, Dieulefait and Wiese)

Let $\ell \geq 5$ be a prime and let V a symplectic $\mathbb{F}_\ell$-vector space of dimension $2d$. Any subgroup G of $\mathsf{GSp}(V)$ which contains a transvection satisfies one of the following:

(I) *There is a non-trivial proper G-stable subspace $W \subset V$.*

(II) *There are non-singular symplectic subspaces $V_i \subset V$ with $i = 1, \ldots, h$, of dimension $2m < 2d$ and a homomorphism $\phi : G \to S_h$ such that $V = \oplus_{i=1}^{h} V_i$ and $\sigma(V_i) = V_{\phi(\sigma)(i)}$ for $\sigma \in G$ and $1 \leq i \leq h$. Moreover, $\phi(G)$ is a transitive subgroup of S_h.*

(III) $\mathsf{Sp}(V) \subseteq G$.

Classification of subgroups of $\mathsf{GSp}_{2d}(\mathbb{F}_\ell)$ with a transvection

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(i) *There is a non-trivial proper G-stable subspace $W \subset V$.* *REDUCIBLE*

(ii) *There are non-singular symplectic subspaces $V_i \subset V$ with $i = 1, \ldots, h$, of dimension $2m < 2d$ and a homomorphism $\phi : G \to S_h$ such that $V = \oplus_{i=1}^{h} V_i$ and $\sigma(V_i) = V_{\phi(\sigma)(i)}$ for $\sigma \in G$ and $1 \leq i \leq h$. Moreover, $\phi(G)$ is a transitive subgroup of S_h.* *INDUCED*

(iii) $\mathsf{Sp}(V) \subseteq G$. *SURJECTIVE*

Idea of the proof

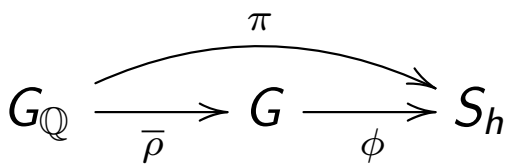
Denote $\overline{\rho} = \overline{\rho}_{A,\ell}$. Let $G = \overline{\rho}(G_{\mathbb{Q}}) \subseteq \mathrm{GSp}_{2d}(\mathbb{F}_\ell)$.

Since G contains a transvection, it is sufficient to show that the induced case does not arise.

Suppose otherwise. Write $V = \oplus_{i=1}^{h} V_i$ where V_i are non-singular symplectic subspaces of dimension $2m < 2d$. Then there is some $\phi : G \to S_h$ with transitive image such that $\sigma(V_i) = V_{\phi(\sigma)(i)}$.
Let

$$G_{\mathbb{Q}} \xrightarrow[\overline{\rho}]{} G \xrightarrow[\phi]{} S_h, \qquad G_{\mathbb{Q}} \xrightarrow{\pi} S_h$$

Idea of the proof



Let $H = \ker(\pi)$. Then $H = G_K$ for some number field $K/\mathbb{Q}$. Moreover, $\overline{\rho}|_{G_K}$ is reducible as the V_i are stable under the action of G_K.

In the proof we show that for $\ell \geq \max(5, d+2)$ the extension $K/\mathbb{Q}$ is unramified at the finite places, and thus K has discriminant ± 1

$\Rightarrow K = \mathbb{Q} \quad \Rightarrow \quad \pi$ is trivial $\Rightarrow$ contradiction

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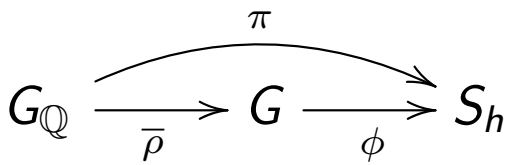

Let $p \neq \ell$ be a prime. As A is semistable, I_p acts unipotently on V, so $\overline{\rho}(\sigma)$ has ℓ-power order for $\sigma \in I_p$.

The order of $\overline{\rho}(\sigma)$ is divisible by the order of $\pi(\sigma)$ which divides h!

As $h = 2d/2m \leq d < \ell \Rightarrow \pi(\sigma) = 1 \Rightarrow K/\mathbb{Q}$ is unramified at p.

The proof for $p = \ell$ is more involved. The bound on ℓ is obtained considering the image of inertia subgroup and applying Raynaud's result.

We now let $A/\mathbb{Q}$ be a **principally polarized abelian threefold**.

Assumptions

(A) A is semistable;

(B) $\ell \geq 5$;

(C) there is a prime q such that the special fibre of the Néron model for A at q has toric dimension 1.

(D) ℓ does not divide $\gcd(\{q \cdot \#\Phi_q : q \in S\})$, where S is the set of primes q satisfying (C) and Φ_q is the group of connected components of the special fibre of the Néron model of A at q.

Under these assumptions the image of $\overline{\rho}_{A,\ell}$ contains a transvection. Then $\overline{\rho}_{A,\ell}$ is either reducible or surjective.

"Algorithm"

Practical method which should, in most cases, produce a small integer B (depending on A) such that for $\ell \nmid B$, the representation $\overline{\rho}_{A,\ell}$ is irreducible and, hence, surjective.

Determinants of Jordan–Hölder factors

Let $\chi : G_{\mathbb{Q}} \to \mathbb{F}_\ell^\times$ denote the mod ℓ cyclotomic character.

We will study the Jordan–Hölder factors W of the $G_{\mathbb{Q}}$-module $A[\ell]$. By the determinant of such a W we mean the determinant of the induced representation $G_{\mathbb{Q}} \to \mathrm{GL}(W)$.

Lemma

Any Jordan–Hölder factor W of the $G_{\mathbb{Q}}$-module $A[\ell]$ has determinant χ^r for some $0 \leq r \leq \dim(W)$.

Weil polynomials

From a prime $p \neq \ell$ of good reduction for A, we will denote by

$$P_p(x) = x^6 + \alpha_p x^5 + \beta_p x^4 + \gamma_p x^3 + p\beta_p x^2 + p^2\alpha_p + p^3 \in \mathbb{Z}[x]$$

the characteristic polynomial of Frobenius $\sigma_p \in G_{\mathbb{Q}}$ at p acting on the Tate module $T_\ell(A)$ (also known as the **Weil polynomial** of A mod p).

The polynomial P_p is independent of ℓ.

Its roots in $\overline{\mathbb{F}}_\ell$ have the form u, v, w, p/u, p/v, p/w.

1-dimensional Jordan–Hölder factors

Let T be a non-empty set of primes of good reduction for A. Let

$$B_1(T) = \gcd(\{p \cdot \#A(\mathbb{F}_p) \ : \ p \in T\}).$$

Lemma

Suppose $\ell \nmid B_1(T)$. The $G_{\mathbb{Q}}$-module $A[\ell]$ does not have any 1-dimensional or 5-dimensional Jordan–Hölder factors.

2-dimensional Jordan–Hölder factors

Lemma

Suppose the $G_{\mathbb{Q}}$-module $A[\ell]$ does not have any 1-dimensional Jordan–Hölder factors, but has either a 2-dimensional or 4-dimensional irreducible subspace U. Then $A[\ell]$ has a 2-dimensional Jordan–Hölder factor W with determinant χ.

Let N be the conductor of A. Let W be a 2-dimensional Jordan–Hölder factor of $A[\ell]$ with determinant χ.
The representation

$$\tau : G_{\mathbb{Q}} \to \mathrm{GL}(W) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$$

is odd (as the determinant is χ), irreducible (as W is a Jordan–Hölder factor) and 2-dimensional.
By Serre's modularity conjecture (Khare, Wintenberger, Dieulefait, Kisin Theorem), this representation is **modular**:

$$\tau \cong \overline{\rho}_{f,\ell}$$

it is equivalent to the mod ℓ representation attached to a newform f of level $M \mid N$ and weight 2.

Let $\mathcal{O}_f$ be the ring of integers of the number field generated by the Hecke eigenvalues of f. Then there is a prime $\lambda \mid \ell$ of $\mathcal{O}_f$ such that for all primes $p \nmid \ell N$,

$$\mathrm{Tr}(\tau(\sigma_p)) \equiv c_p(f) \pmod{\lambda}$$

where $\sigma_p \in G_{\mathbb{Q}}$ is a Frobenius element at p and $c_p(f)$ is the p-th Hecke eigenvalue of f.

As W is a Jordan–Hölder factor of $A[\ell]$ we see that $x^2 - c_p(f)x + p$ is a factor modulo λ of P_p.

Now let $H_{M,p}$ be the p-th Hecke polynomial for the new subspace $S_2^{\text{new}}(M)$ of cusp forms of weight 2 and level M. This has the form

$$H_{M,p} = \prod (x - c_p(g)),$$

where g runs through the newforms of weight 2 and level M. Write

$$H'_{M,p}(x) = x^d H_{M,p}(x + p/x) \in \mathbb{Z}[x],$$

where $d = \deg(H_{M,p}) = \dim(S_2^{\text{new}}(M))$.

It follows that $x^2 - c_p(f)x + p$ divides $H'_{M,p}$.

Let

$$R(M,p) = \mathrm{Res}(P_p, H'_{M,p}) \in \mathbb{Z}\,,$$

where Res denotes resultant. If $R(M,p) \neq 0$ then we have a bound on ℓ.

The integers $R(M,p)$ can be very large. Given a non-empty set T of rational primes p of good reduction for A, let

$$R(M,T) = \gcd(\{p \cdot R(M,p) \ : \ p \in T\}).$$

In practice, we have found that for a suitable choice of T, the value $R(M,T)$ is fairly small.

Let

$$B_2'(T) = \mathrm{lcm}(R(M, T))$$

where M runs through the divisors of N such that $\dim(S_2^{\mathrm{new}}(M)) \neq 0$, and let

$$B_2(T) = \mathrm{lcm}(B_1(T), B_2'(T))$$

where $B_1(T)$ is given as before.

Lemma

Let T be a non-empty set of rational primes of good reduction for A, and suppose $\ell \nmid B_2(T)$. Then $A[\ell]$ does not have 1-dimensional Jordan–Hölder factors, and does not have irreducible 2- or 4-dimensional subspaces.

We fail to bound ℓ in the above lemma if $R(M, p) = 0$ for all primes p of good reduction.

Here are two situations where this can happen:

- $A \cong_{\mathbb{Q}} E \times A'$ where E is an elliptic curve and A' an abelian surface.
- A is of GL_2-type.

Note that in both these situations $\mathrm{End}_{\overline{\mathbb{Q}}}(A) \neq \mathbb{Z}$.

We expect, but are unable to prove, that if $\mathrm{End}_{\overline{\mathbb{Q}}}(A) = \mathbb{Z}$ then there will be primes p such that $R(M, p) \neq 0$.

Similarly, it is possible to give bounds B_3, B_4 to rule out 3-dimensional Jordan–Hölder factors.

Theorem (A., Lemos and Siksek)

Let A and ℓ satisfy conditions (A)–(D). Let T be a non-empty set of primes of good reduction for A. Let

$$B(T) = \mathrm{lcm}(B_2(T), B_3(T), B_4(T)).$$

If $\ell \nmid B(T)$ then $\overline{\rho}_{A,\ell}$ is surjective.

Example

Theorem (A., Lemos and Siksek)

Let $C/\mathbb{Q}$ be the following genus 3 hyperelliptic curve,

$$C : y^2 + (x^4 + x^3 + x + 1)y = x^6 + x^5.$$

and write J for its Jacobian. Then

$$\overline{\rho}_{J,\ell}(G_{\mathbb{Q}}) = \mathsf{GSp}_6(\mathbb{F}_\ell)$$

for all odd prime ℓ.

Proof.

For $\ell \geq 5$ we apply the algorithm with $T = \{2, 5, 7\}$. For $\ell = 3$, we prove the result by direct computations. □

Generalizations...

- ... over **number fields**: obstruction coming from the Weil pairing, e.g.

$$E : y^2 + (\frac{\sqrt{101}+1}{2})y = x^3 + x^2 - 2x - 7 \quad \text{over } \mathbb{Q}(\sqrt{101})$$

$$\overline{\rho}_{E,\ell}(\mathsf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{101}))) \cong \mathsf{GL}_2(\mathbb{F}_\ell) \quad \forall \text{ prime } \ell \neq 101$$

$$\rho_{E,101}(\mathsf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{101}))) \subseteq D \cdot \mathsf{SL}_2(\mathbb{F}_{101})$$

where D is the set of invertible squares in $\mathbb{F}_{101}$.

Generalizations...

- ... to **higher genus**: work in progress with Vladimir Dokchitser.

Let $C : y^2 = f(x)$ be a **hyperelliptic curve** of genus g over $\mathbb{Q}$, where $f \in \mathbb{Z}[x]$ is a squarefree polynomial of degree $2g + 2$. Let $J := \mathrm{Jac}(C)$.

Idea

Prescribe congruence conditions on $f(x)$ to have certain inertia elements in the representation and obtain uniform realization.

Generalization to higher genus

Result: realization of $\mathrm{GSp}_{2g}(\mathbb{F}_\ell)$ as Galois group over $\mathbb{Q}$.

For each g there exists a **positive density** of polynomial $f(x) \in \mathbb{Z}[x]$ such that **simultaneously** for all odd primes ℓ the mod ℓ Galois representation associated to the Jacobian J of the hyperelliptic curve $y^2 = f(x)$ is surjective, i.e. $\mathrm{Gal}(\mathbb{Q}(\mathrm{Jac}(y^2 = f(x))[\ell]/\mathbb{Q}) \cong \mathrm{GSp}_{2g}(\mathbb{F}_\ell)$.

Remark: when g is large enough we assume the Goldbach conjecture "twice" and more, i.e. we assume that there exist primes q_1, q_2, q_3, q_4, q_5 such that

$$2g + 2 = q_1 + q_2 = q_4 + q_5,$$

$$2g + 2 > q_3 > q_5 > q_2 \geq q_1 > q_4.$$

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Thanks!