Abelian varieties and the inverse Galois problem

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Galois representations may answer the inverse Galois problem for finite linear groups.

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ and let $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

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Let A be a principally polarized abelian variety over $\mathbb{Q}$ of dimension d.

Let ℓ be a prime and $A[\ell]$ the ℓ-torsion subgroup:

$$A[\ell] := \{P \in A(\overline{\mathbb{Q}}) \mid [\ell]P = 0\} \cong (\mathbb{Z}/\ell\mathbb{Z})^{2d}.$$

$A[\ell]$ is a $2d$-dimensional $\mathbb{F}_\ell$-vector space, as well as a $G_{\mathbb{Q}}$-module.

The polarization induces a symplectic pairing, the mod ℓ Weil pairing on $A[\ell]$, which is a bilinear, alternating, non-degenerate pairing:

$$\langle \ , \ \rangle \ : \ A[\ell] \times A[\ell] \to \mu_\ell$$

that is Galois invariant: $\forall \sigma \in G_{\mathbb{Q}}, \ \forall v, w \in A[\ell]$

$$\langle \sigma v, \sigma w \rangle = \chi(\sigma) \langle v, w \rangle,$$

where $\chi : G_{\mathbb{Q}} \to \mathbb{F}_\ell^\times$ is the mod ℓ cyclotomic character.

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$$\overline{\rho}_{A,\ell} : G_{\mathbb{Q}} \to \mathrm{GSp}(A[\ell], \langle\ ,\ \rangle) \cong \mathrm{GSp}_{2d}(\mathbb{F}_\ell).$$

Theorem (Serre)

Let A be a principally polarized abelian variety of dimension d, defined over $\mathbb{Q}$. Assume that $d = 2, 6$ or d is odd and, furthermore, assume that $\mathrm{End}_{\overline{\mathbb{Q}}}(A) = \mathbb{Z}$. Then there exists a bound B_A such that for all primes $\ell > B_A$ the representation $\overline{\rho}_{A,\ell}$ is surjective.

The conclusion of the theorem is known to be false for general d (counterexample by Mumford for $d = 4$).

Open question

Given d as in the theorem, is there a uniform bound B_d depending only on d, such that for all principally polarized abelian varieties A over $\mathbb{Q}$ of dimension d with $\mathrm{End}_{\overline{\mathbb{Q}}}(A) = \mathbb{Z}$, and all $\ell > B_d$, the representation $\overline{\rho}_{A,\ell}$ is surjective?

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Much easier for semistable elliptic curves:

Theorem (Serre)

Let $E/\mathbb{Q}$ be a semistable elliptic curve, and $\ell \geq 11$ be a prime. Then $\overline{\rho}_{E,\ell}$ is surjective.

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The Galois representation attached to the ℓ-torsion of the elliptic curve $y^2 + y = x^3 - x$ is surjective for all prime ℓ. This gives a realization $\mathrm{GL}_2(\mathbb{F}_\ell)$ as Galois group for all ℓ.

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Let C be the genus 2 hyperelliptic curve given by $y^2 = x^5 - x + 1$ and let J denotes its Jacobian. Dieulefait proved that $\overline{\rho}_{J,\ell}$ is surjective for all odd prime ℓ. This gives a realization $\mathrm{GSp}_4(\mathbb{F}_\ell)$ as Galois group for all odd ℓ.

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$\mathrm{GSp}_6(\mathbb{F}_\ell)$

What about genus 3 curves?

Theorem (A., Lemos and Siksek)

Let A be a semistable principally polarized abelian variety of dimension $d \geq 1$ over $\mathbb{Q}$ and let $\ell \geq \max(5, d+2)$ be prime.
Suppose the image of $\overline{\rho}_{A,\ell} \; : \; G_{\mathbb{Q}} \to \mathrm{GSp}_{2d}(\mathbb{F}_\ell)$ contains a transvection.
Then $\overline{\rho}_{A,\ell}$ is either reducible or surjective.

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Transvection

Definition

Let $(V, \langle\ ,\ \rangle)$ be a finite-dimensional symplectic vector space over $\mathbb{F}_\ell$. A **transvection** *is an element $T \in \mathrm{GSp}(V, \langle\ ,\ \rangle)$ which fixes a hyperplane $H \subset V$.*

Therefore, a transvection is a unipotent element $\sigma \in \mathrm{GSp}(V, \langle\ ,\ \rangle)$ such that $\sigma - I$ has rank 1.

When does $\overline{\rho}_{A,\ell}(G_{\mathbb{Q}})$ contain a transvection?

Let $q \neq \ell$ be a prime and suppose that the following two conditions are satisfied:

- the special fibre of the Néron model for A at q has toric dimension 1;
- $\ell \nmid \#\Phi_q$, where Φ_q is the group of connected components of the special fibre of the Néron model at q.

Then the image of $\overline{\rho}_{A,\ell}$ contains a transvection (Hall).

When does $\overline{\rho}_{A,\ell}(G_{\mathbb{Q}})$ contain a transvection?

Let $C/\mathbb{Q}$ be a hyperelliptic curve of genus d:

$$C : y^2 = f(x)$$

where $f \in \mathbb{Z}[x]$ is a squarefree polynomial.

Let p be an odd prime not dividing the leading coefficient of f such that f modulo p has one root in $\overline{\mathbb{F}}_p$ having multiplicity precisely 2, with all other roots simple.
Then the Néron model of the Jacobian at p has toric dimension 1 (Hall).

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In the proof of this theorem we rely on:

- the classification due to Arias-de-Reyna, Dieulefait and Wiese of subgroups of $\mathrm{GSp}_{2d}(\mathbb{F}_\ell)$ containing a transvection;
- results of Raynaud on the image of the inertia subgroup.

Classification of subgroups of $\mathrm{GSp}_{2d}(\mathbb{F}_\ell)$ with a transvection

Theorem (Arias-de-Reyna, Dieulefait and Wiese)

Let $\ell \geq 5$ be a prime and let V a symplectic $\mathbb{F}_\ell$-vector space of dimension $2d$. Any subgroup G of $\mathrm{GSp}(V)$ which contains a transvection satisfies one of the following:

- (I) *There is a non-trivial proper G-stable subspace $W \subset V$.*
- (II) *There are non-singular symplectic subspaces $V_i \subset V$ with $i = 1, \ldots, h$, of dimension $2m < 2d$ and a homomorphism $\phi : G \to S_h$ such that $V = \oplus_{i=1}^{h} V_i$ and $\sigma(V_i) = V_{\phi(\sigma)(i)}$ for $\sigma \in G$ and $1 \leq i \leq h$. Moreover, $\phi(G)$ is a transitive subgroup of S_h.*
- (III) $\mathrm{Sp}(V) \subseteq G$.

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We apply this to $G = \overline{\rho}_{A,\ell}(G_{\mathbb{Q}})$ where A and ℓ are as in the main theorem. If $\mathrm{Sp}_{2d}(\mathbb{F}_\ell) \subseteq G$ then $G = \mathrm{GSp}_{2d}(\mathbb{F}_\ell)$, since the mod ℓ cyclotomic character is surjective.

Inertia and a theorem of Raynaud

Theorem (Raynaud)

Let A be an abelian variety over $\mathbb{Q}$. Let ℓ be a prime of semistable reduction for A. Regard $A[\ell]$ as an I_ℓ-module and let V be a Jordan-Hölder factor of dimension n over $\mathbb{F}_\ell$. Let $\psi_n : I_\ell \to \mathbb{F}_{\ell^n}^\times$ be a fundamental character of level n. Then V has the structure of a 1-dimensional $\mathbb{F}_{\ell^n}$-vector space and the action of I_ℓ on it is given by a character $\varpi : I_\ell \to \mathbb{F}_{\ell^n}^\times$, where $\varpi = \psi_n^{\sum_{i=0}^{n-1} a_i \ell^i}$ with $a_i = 0$ or 1.

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Suppose otherwise. Write $V = \oplus_{i=1}^{h} V_i$ where V_i are non-singular symplectic subspaces of dimension $2m < 2d$. Then there is some $\phi : G \to S_h$ with transitive image such that $\sigma(V_i) = V_{\phi(\sigma)(i)}$.

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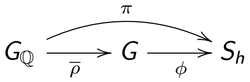

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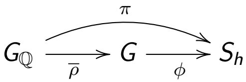

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$$\begin{array}{c} \overset{\pi}{\curvearrowright} \\ G_{\mathbb{Q}} \xrightarrow[\overline{\rho}]{} G \xrightarrow[\phi]{} S_h \end{array}$$

Let $H = \ker(\pi)$. Then $H = G_K$ for some number field $K/\mathbb{Q}$. Moreover, $\overline{\rho}|_{G_K}$ is reducible as the V_i are stable under the action of G_K.

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In the proof we show that for $\ell \geq \max(5, d+2)$ the extension $K/\mathbb{Q}$ is unramified at the finite places, and thus K has discriminant ± 1

$\Rightarrow K = \mathbb{Q} \quad \Rightarrow \quad \pi$ is trivial $\Rightarrow$ contradiction

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Let $p \neq \ell$ be a prime. As A is semistable, I_p acts unipotently on V, so $\overline{\rho}(\sigma)$ has ℓ-power order for $\sigma \in I_p$.

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The proof for $p = \ell$ is more involved. The bound on ℓ is obtained considering the image of inertia subgroup and applying Raynaud's result.

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(D) ℓ does not divide $\gcd(\{q \cdot \#\Phi_q : q \in S\})$, where S is the set of primes q satisfying (C) and Φ_q is the group of connected components of the special fibre of the Néron model of A at q.

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"Algorithm"

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We will apply this procedure to the Jacobian of the hyperelliptic curve:

$$C \;:\; y^2 + (x^4 + x^3 + x + 1)y = x^6 + x^5.$$

Determinants of Jordan–Hölder factors

Let $\chi : G_{\mathbb{Q}} \to \mathbb{F}_\ell^\times$ denote the mod ℓ cyclotomic character.

We will study the Jordan–Hölder factors W of the $G_{\mathbb{Q}}$-module $A[\ell]$. By the determinant of such a W we mean the determinant of the induced representation $G_{\mathbb{Q}} \to \mathrm{GL}(W)$.

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Lemma

Any Jordan–Hölder factor W of the $G_{\mathbb{Q}}$-module $A[\ell]$ has determinant χ^r for some $0 \leq r \leq \dim(W)$.

Weil polynomials

From a prime $p \neq \ell$ of good reduction for A, we will denote by

$$P_p(x) = x^6 + \alpha_p x^5 + \beta_p x^4 + \gamma_p x^3 + p\beta_p x^2 + p^2\alpha_p + p^3 \in \mathbb{Z}[x]$$

the characteristic polynomial of Frobenius $\sigma_p \in G_{\mathbb{Q}}$ at p acting on the Tate module $T_\ell(A)$ (also known as the **Weil polynomial** of A mod p).

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the characteristic polynomial of Frobenius $\sigma_p \in G_{\mathbb{Q}}$ at p acting on the Tate module $T_\ell(A)$ (also known as the **Weil polynomial** of A mod p).
The polynomial P_p is independent of ℓ.
Its roots in $\overline{\mathbb{F}}_\ell$ have the form u, v, w, p/u, p/v, p/w.

1-dimensional Jordan–Hölder factors

Let T be a non-empty set of primes of good reduction for A. Let

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Lemma

Suppose $\ell \nmid B_1(T)$. The $G_{\mathbb{Q}}$-module $A[\ell]$ does not have any 1-dimensional or 5-dimensional Jordan–Hölder factors.

2-dimensional Jordan–Hölder factors

Lemma

Suppose the $G_{\mathbb{Q}}$-module $A[\ell]$ does not have any 1-dimensional Jordan–Hölder factors, but has either a 2-dimensional or 4-dimensional irreducible subspace U. Then $A[\ell]$ has a 2-dimensional Jordan–Hölder factor W with determinant χ.

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is odd (as the determinant is χ), irreducible (as W is a Jordan–Hölder factor) and 2-dimensional. By Serre's modularity conjecture (Khare, Wintenberger, Dieulefait, Kisin Theorem), this representation is **modular**:

$$\tau \cong \overline{\rho}_{f,\ell}$$

it is equivalent to the mod ℓ representation attached to a newform f of level $M \mid N$ and weight 2.

Let $\mathcal{O}_f$ be the ring of integers of the number field generated by the Hecke eigenvalues of f. Then there is a prime $\lambda \mid \ell$ of $\mathcal{O}_f$ such that for all primes $p \nmid \ell N$,

$$\mathrm{Tr}(\tau(\sigma_p)) \equiv c_p(f) \pmod{\lambda}$$

where $\sigma_p \in G_{\mathbb{Q}}$ is a Frobenius element at p and $c_p(f)$ is the p-th Hecke eigenvalue of f.

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As W is a Jordan–Hölder factor of $A[\ell]$ we see that $x^2 - c_p(f)x + p$ is a factor modulo λ of P_p.

Now let $H_{M,p}$ be the p-th Hecke polynomial for the new subspace $S_2^{\mathrm{new}}(M)$ of cusp forms of weight 2 and level M. This has the form

$$H_{M,p} = \prod (x - c_p(g)),$$

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where g runs through the newforms of weight 2 and level M.Write

$$H'_{M,p}(x) = x^d H_{M,p}(x + p/x) \in \mathbb{Z}[x],$$

where $d = \deg(H_{M,p}) = \dim(S_2^{\mathrm{new}}(M))$.

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where $d = \deg(H_{M,p}) = \dim(S_2^{\mathrm{new}}(M))$.

It follows that $x^2 - c_p(f)x + p$ divides $H'_{M,p}$.

Let

$$R(M,p) = \mathrm{Res}(P_p, H'_{M,p}) \in \mathbb{Z}\,,$$

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The integers $R(M,p)$ can be very large. Given a non-empty set T of rational primes p of good reduction for A, let

$$R(M,T) = \gcd(\{p \cdot R(M,p) \;:\; p \in T\}).$$

In practice, we have found that for a suitable choice of T, the value $R(M,T)$ is fairly small.

Let

$$B_2'(T) = \mathrm{lcm}(R(M, T))$$

where M runs through the divisors of N such that $\dim(S_2^{\mathrm{new}}(M)) \neq 0$, and let

$$B_2(T) = \mathrm{lcm}(B_1(T), B_2'(T))$$

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Lemma

Let T be a non-empty set of rational primes of good reduction for A, and suppose $\ell \nmid B_2(T)$. Then $A[\ell]$ does not have 1-dimensional Jordan–Hölder factors, and does not have irreducible 2- or 4-dimensional subspaces.

We fail to bound ℓ in the above lemma if $R(M, p) = 0$ for all primes p of good reduction.

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- Suppose $A \cong_{\mathbb{Q}} E \times A'$ where E is an elliptic curve and A' an abelian surface. Let $M \mid N$ be the conductor of the elliptic curve, and f to be the newform associated to E by modularity, then $x^2 - c_p(f)x + p$ is a factor of $P_p(x) \Rightarrow R(M, p) = 0$ for all $p \nmid N$.

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- Suppose $A \cong_{\mathbb{Q}} E \times A'$ where E is an elliptic curve and A' an abelian surface. Let $M \mid N$ be the conductor of the elliptic curve, and f to be the newform associated to E by modularity, then $x^2 - c_p(f)x + p$ is a factor of $P_p(x) \Rightarrow R(M, p) = 0$ for all $p \nmid N$.
- Suppose A is of GL_2-type. Let f be the corresponding eigenform, then again $x^2 - c_p(f)x + p$ is a factor of $P_p(x)$ in $\mathcal{O}_f[x]$ $\Rightarrow R(M, p) = 0$ for all $p \nmid N$.

Note that in both these situations $\mathrm{End}_{\overline{\mathbb{Q}}}(A) \neq \mathbb{Z}$.

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We expect, but are unable to prove, that if $\mathrm{End}_{\overline{\mathbb{Q}}}(A) = \mathbb{Z}$ then there will be primes p such that $R(M, p) \neq 0$.

3-dimensional Jordan–Hölder factors

Lemma

Suppose $A[\ell]$ has Jordan–Hölder filtration $0 \subset U \subset A[\ell]$ where both U and $A[\ell]/U$ are irreducible and 3-dimensional. Moreover, let u_1, u_2, u_3 be a basis for U, and let

$$G_{\mathbb{Q}} \to \mathrm{GL}_3(\mathbb{F}_\ell), \qquad \sigma \mapsto M(\sigma)$$

give the action of $G_{\mathbb{Q}}$ on U with respect to this basis. Then we can extend u_1, u_2, u_3 to a symplectic basis $u_1, u_2, u_3, w_1, w_2, w_3$ for $A[\ell]$ so that the action of $G_{\mathbb{Q}}$ on $A[\ell]$ with respect to this basis is given by

$$G_{\mathbb{Q}} \to \mathrm{GSp}_6(\mathbb{F}_\ell), \qquad \sigma \mapsto \left(\begin{array}{c|c} M(\sigma) & * \\ \hline \mathbf{0} & \chi(\sigma)(M(\sigma)^t)^{-1} \end{array}\right).$$

$\det(U) = \chi^r$ and $\det(A[\ell]/U) = \chi^s$ where $0 \leq r, s \leq 3$ with $r + s = 3$.

Lemma

Let p be a prime of good reduction for A. For ease write α, β and γ for the coefficients α_p, β_p, γ_p in the equation of the Weil polynomial. Suppose $p+1 \neq \alpha$. Let

$$\delta = \frac{-p^2\alpha + p^2 + p\alpha^2 - p\alpha - p\beta + p - \beta + \gamma}{(p-1)(p+1-\alpha)} \in \mathbb{Q}, \qquad \epsilon = \delta + \alpha \in \mathbb{Q}.$$

Let $g(x) = (x^3 + \epsilon x^2 + \delta x - p)(x^3 - \delta x^2 - p\epsilon x - p^2) \in \mathbb{Q}[x]$. Write k for the greatest common divisor of the numerators of the coefficients in $P_p - g$. Let

$$K_p = p(p-1)(p+1-\alpha)k.$$

Then $K_p \neq 0$. Moreover, if $\ell \nmid K_p$ then $A[\ell]$ does not have a Jordan–Hölder filtration as in the previous Lemma with $\det(U) = \chi$ or χ^2.

Lemma

Let p be a prime of good reduction for A. Write α, β and γ for the coefficients α_p, β_p, γ_p in the equation of the Weil polynomial. Suppose $p^3 + 1 \neq p\alpha$. Let $\epsilon' = p\delta' + \alpha \in \mathbb{Q}$ where

$$\delta' = \frac{-p^5\alpha + p^4 + p^3\alpha^2 - p^3\beta - p^2\alpha + p\gamma + p - \beta}{(p^3 - 1)(p^3 + 1 - p\alpha)} \in \mathbb{Q}\,.$$

Let $g'(x) = (x^3 + \epsilon' x^2 + \delta' x - 1)(x^3 - p\delta' x^2 - p^2\epsilon' x - p^3) \in \mathbb{Q}[x]$. Write k' for the greatest common divisor of the numerators of the coefficients in $P_p - g'$. Let

$$K'_p = p(p^3 - 1)(p^3 + 1 - p\alpha)k'.$$

Then $K'_p \neq 0$. Moreover, if $\ell \nmid K'_p$ then $A[\ell]$ does not have a Jordan–Hölder filtration as in the above Lemma with $\det(U) = 1$ or χ^3.

Summary

The following theorem summarizes all the lemmas:

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Theorem (A., Lemos and Siksek)

Let A and ℓ satisfy conditions (A)–(D). Let T be a non-empty set of primes of good reduction for A. Let

$$B_3(T) = \gcd(\{K_p \,:\, p \in T\}), \qquad B_4(T) = \gcd(\{K'_p \,:\, p \in T\}),$$

where K_p and K'_p are defined in the last two Lemmas. Let

$$B(T) = \operatorname{lcm}(B_2(T), B_3(T), B_4(T)).$$

If $\ell \nmid B(T)$ then $\overline{\rho}_{A,\ell}$ is surjective.

Example

Theorem (A., Lemos and Siksek)

Let $C/\mathbb{Q}$ be the following genus 3 hyperelliptic curve,

$$C \ : \ y^2 + (x^4 + x^3 + x + 1)y = x^6 + x^5.$$

and write J for its Jacobian. Then

$$\overline{\rho}_{J,\ell}(G_{\mathbb{Q}}) = \mathrm{GSp}_6(\mathbb{F}_\ell)$$

for all odd prime ℓ.

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Proof.

For $\ell \geq 5$ we apply the algorithm.

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Proof.

For $\ell \geq 5$ we apply the algorithm.For $\ell = 3$, we prove the result by direct computations. □

Future generalization

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- over **number fields**: obstruction coming from the Weil pairing, e.g.

$$E : y^2 + (\frac{\sqrt{101}+1}{2})y = x^3 + x^2 - 2x - 7 \quad \text{over } \mathbb{Q}(\sqrt{101})$$

$$\overline{\rho}_{E,\ell}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{101}))) \cong \mathrm{GL}_2(\mathbb{F}_\ell) \quad \forall \text{ prime } \ell \neq 101$$

$$\rho_{E,101}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{101}))) \subseteq D \cdot \mathrm{SL}_2(\mathbb{F}_{101})$$

where D is the set of invertible squares in $\mathbb{F}_{101}$.

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$$\rho_{E,101}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{101}))) \subseteq D \cdot \mathrm{SL}_2(\mathbb{F}_{101})$$

  where D is the set of invertible squares in $\mathbb{F}_{101}$.

- to **higher genus**: work in progress with Vladimir Dokchitser. Result: realization of $\mathrm{GSp}_{2g}(\mathbb{F}_\ell)$ as Galois group over $\mathbb{Q}$ for every $g \in \mathbb{Z}_{>0}$ and for every odd prime ℓ using hyperelliptic curves.

Abelian varieties and the inverse Galois problem

Samuele Anni
joint with Pedro Lemos and Samir Siksek

University of Warwick

Number Theory Seminar,
UCSD, 19th May 2016

Thanks!