Twists and residual modular Galois representations

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Let us fix a positive integer $n \in \mathbb{Z}_{>0}$.

Definition

The **congruence subgroup** $\Gamma_1(n)$ of $SL_2(\mathbb{Z})$ is the subgroup given by

$$\Gamma_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : n \mid a-1\,,\ n \mid c \right\}.$$

The integer n is called **level** of the congruence subgroup.

Over the upper half plane:

$$\mathbb{H} = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$$

we can define an action of $\Gamma_1(n)$ via **fractional transformations**:

$$\begin{aligned} \Gamma_1(n) \times \mathbb{H} &\to \mathbb{H} \\ (\gamma, z) &\mapsto \gamma(z) = \frac{az+b}{cz+d} \end{aligned}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Moreover, if $n \geq 4$ then $\Gamma_1(n)$ acts freely on $\mathbb{H}$.

Escher, Reducing Lizards Tessellation

Definition

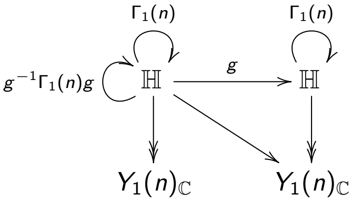
We define the **modular curve** $Y_1(n)_{\mathbb{C}}$ to be the non-compact Riemann surface obtained giving on $\Gamma_1(n)\backslash\mathbb{H}$ the complex structure induced by the quotient map. Let $X_1(n)_{\mathbb{C}}$ be the compactification of $Y_1(n)_{\mathbb{C}}$.

Fact: $Y_1(n)_{\mathbb{C}}$ can be defined algebraically over $\mathbb{Q}$ (in fact over $\mathbb{Z}[1/n]$).

The group $GL_2^+(\mathbb{Q})$ acts on $\mathbb{H}$ via fractional transformation, and its action has a particular behaviour with respect to $\Gamma_1(n)$.

Proposition

For every $g \in GL_2^+(\mathbb{Q})$, the discrete groups $g\Gamma_1(n)g^{-1}$ and $\Gamma_1(n)$ are commensurable



We define operators on $Y_1(n)$ through the correspondences given before:

- the **Hecke operators** T_p for every prime p, using $g = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \in GL_2^+(\mathbb{Q})$;
- the **diamond operators** $\langle d \rangle$ for every $d \in (\mathbb{Z}/n\mathbb{Z})^*$, using $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(n)$, where $\Gamma_0(n)$ is the set of matrices in $\mathrm{SL}_2(\mathbb{Z})$ which are upper triangular modulo n.

For $n \geq 5$ and k positive integers, let ℓ be a prime not dividing n. Following Katz, we define the space of mod ℓ cusp forms as

mod ℓ cusp forms

$$S(n,k)_{\overline{\mathbb{F}}_\ell} = \mathrm{H}^0(X_1(n)_{\overline{\mathbb{F}}_\ell}, \omega^{\otimes k}(-\,\mathrm{Cusps})).$$

$S(n,k)_{\overline{\mathbb{F}}_\ell}$ is a finite dimensional $\overline{\mathbb{F}}_\ell$-vector space, equipped with Hecke operators T_n $(n \geq 1)$ and diamond operators $\langle d \rangle$ for every $d \in (\mathbb{Z}/n\mathbb{Z})^*$.

Analogous definition in characteristic zero and over any ring where n is invertible.

One may think that mod ℓ modular forms come from reduction of characteristic zero modular forms mod ℓ:

$$S(n,k)_{\mathbb{Z}[1/n]} \to S(n,k)_{\mathbb{F}_\ell}.$$

Unfortunately, this map is **not surjective** for $k=1$.

Even worse: given a character $\epsilon\colon (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}^*$ the map

$$S(n,k,\epsilon)_{\mathcal{O}_K} \to S(n,k,\overline{\epsilon})_{\mathbb{F}}$$

is **not** always **surjective** even if $k>1$.
$\mathcal{O}_K$ is the ring of integers of the number field where ϵ is defined, $\mathbb{F}$ is the residue field at ℓ and $\overline{\epsilon}$ is the reduction of ϵ
$S(n,k,\epsilon)_{\mathcal{O}_K} = \{f \in S(n,k)_{\mathcal{O}_K} | \forall d \in (\mathbb{Z}/n\mathbb{Z})^*, \langle d\rangle f = \epsilon(d) f\}$.

Definition

The **Hecke algebra** $\mathbb{T}(n,k)$ of $S(n,k)_{\mathbb{C}}$ is the $\mathbb{Z}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(S(\Gamma_1(n),k)_{\mathbb{C}})$ generated by Hecke operators T_p for every prime p and by diamond operators $\langle d \rangle$ for every $d \in (\mathbb{Z}/n\mathbb{Z})^*$.

Fact:

$\mathbb{T}(n,k)$ is finitely generated as $\mathbb{Z}$-module.

Given a character $\epsilon\colon (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}^*$, we associate a Hecke algebra $\mathbb{T}_\epsilon(n,k)$ to each $S(n,k,\epsilon)_{\mathbb{C}}$:

$$S(n,k,\epsilon)_{\mathbb{C}} = \{f \in S(n,k)_{\mathbb{C}} | \forall d \in (\mathbb{Z}/n\mathbb{Z})^*, \langle d \rangle f = \epsilon(d) f\}.$$

Theorem (Deligne, Shimura)

Let n and k be positive integers. Let $\mathbb{F}$ be a finite field of characteristic ℓ, with ℓ not dividing n, and $f : \mathbb{T}(n,k) \twoheadrightarrow \mathbb{F}$ a surjective morphism of rings. Then there is a continuous semi-simple representation:

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}),$$

unramified outside $n\ell$, such that for all p not dividing $n\ell$ we have:

$$\mathrm{Trace}(\rho_f(\mathrm{Frob}_p)) = f(T_p) \text{ and } \det(\rho_f(\mathrm{Frob}_p)) = f(\langle p \rangle)p^{k-1} \text{ in } \mathbb{F}.$$

Such a ρ_f is unique up to isomorphism.

Computing ρ_f is “difficult”, but theoretically it **can be done in polynomial time** in $n, k, \#\mathbb{F}$:

Edixhoven, Couveignes, de Jong, Merkl, Bruin, Bosman ($\#\mathbb{F} \leq 32$);
Mascot, Zeng, Tian ($\#\mathbb{F} \leq 41$).

Question

Can we compute the image of a residual modular Galois representation without computing the representation?

Theorem (A.)

There is a polynomial time algorithm which takes as input:

- *n and k be positive integers;*
- *ℓ be a prime number not dividing n, such that $2 \leq k \leq \ell + 1$;*
- *a character $\epsilon : (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}^*$;*
- *a morphism of ring $f : \mathbb{T}_\epsilon(n, k) \to \overline{\mathbb{F}}_\ell$, and in particular the images of all diamond operators and of the T_p operators up to a bound B,*

and gives as output the image of the associated Galois representation ρ_f, up to conjugacy as subgroup of $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ without computing ρ_f.

In almost all cases, the bound B is the Sturm Bound for $\Gamma_0(n)$ and weight k:

$$\frac{k}{12} \cdot n \cdot \prod_{p|n \text{ prime}} \left(1 + \frac{1}{p}\right) \ll \frac{k}{12} \cdot n \log \log n$$

In the cases when this bound is not enough, then the Sturm Bound for $\Gamma_0(nq^2)$ and weight k, where q is the smallest odd prime not dividing n, is the required bound.

Some of the problems studied:

- ρ_f can arise from lower level or weight, i.e. there exists $g \in S(m,j)_{\overline{\mathbb{F}}_\ell}$ with $m \leq n$ or $j \leq k$ such that $\rho_g \cong \rho_f$
- ρ_f can arise as twist of a representation of lower conductor, i.e. there exist $g \in S(m,j)_{\overline{\mathbb{F}}_\ell}$ with $m \leq n$ or $j \leq k$ and a Dirichlet character χ such that $\rho_g \otimes \chi \cong \rho_f$

One of the principal ingredients:

Theorem (Khare, Wintenberger, Dieulefait, Kisin), Serre's Conjecture

Let ℓ be a prime number and let $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be an odd, absolutely irreducible, continuous representation. Then ρ is **modular** of level $N(\rho)$, weight $k(\rho)$ and character $\epsilon(\rho)$.

- $N(\rho)$ (the level) is the Artin conductor away from ℓ.
- $k(\rho)$ (the weight) is given by a recipe in terms of $\rho|_{I_\ell}$.
- $\epsilon(\rho)\colon (\mathbb{Z}/N(\rho)\mathbb{Z})^* \to \overline{\mathbb{F}}_\ell^*$ is given by:

$$\det \circ \rho = \epsilon(\rho)\chi^{k(\rho)-1}.$$

Setting $(*)$

- n and k be positive integers;
- ℓ be a prime number not dividing n, such that $2 \leq k \leq \ell + 1$;
- $\epsilon : (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}^*$ be a character;
- $f : \mathbb{T}_\epsilon(n, k) \to \overline{\mathbb{F}}_\ell$ be a morphism of rings;
- $\rho_f : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$ be the unique, up to isomorphism, continuous semi-simple representation attached to f;
- $\overline{\epsilon} : (\mathbb{Z}/n\mathbb{Z})^* \to \overline{\mathbb{F}}_\ell^*$ be the character defined by $\overline{\epsilon}(a) = f(\langle a \rangle)$ for all $a \in (\mathbb{Z}/n\mathbb{Z})^*$.

Let p be a prime dividing $n\ell$. Let us denote by

- $G_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \subset G_\mathbb{Q}$ the decomposition subgroup at p;
- I_p the inertia subgroup, I_t the tame inertia subgroup;
- $G_{i,p}$, with $i \in \mathbb{Z}_{>0}$, the higher ramification subgroups ($I_p = G_{0,p}$).

Notation: given a residual representation ρ, we will denote as $\mathrm{N}_p(\rho)$ the valuation at p of the Artin conductor of ρ.

Local representation at ℓ

Theorem (Deligne)

Assume setting $(*)$. Suppose that $f(T_\ell) \neq 0$. Then $\rho_f|_{G_\ell}$ is reducible, and up to conjugation in $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$, we have

$$\rho_f|_{G_\ell} \cong \begin{pmatrix} \chi_\ell^{k-1}\lambda(\overline{\epsilon}(\ell)/f(T_\ell)) & * \\ 0 & \lambda(f(T_\ell)) \end{pmatrix}$$

where $\lambda(a)$ is the unramified character of G_ℓ taking $\mathrm{Frob}_\ell \in G_\ell/I_\ell$ to a, for any $a \in \overline{\mathbb{F}}_\ell^*$.

Theorem (Fontaine)

Assume setting (∗). Suppose that $f(T_\ell) = 0$. Then $\rho_f|_{G_\ell}$ is irreducible, and up to conjugation in $\mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$, we have

$$\rho_f|_{I_\ell} \cong \begin{pmatrix} \varphi'^{k-1} & 0 \\ 0 & \varphi^{k-1} \end{pmatrix}$$

where $\varphi', \varphi\colon I_t \to \overline{\mathbb{F}}_\ell^*$ are the two fundamental characters of level 2.

Local representation at primes dividing the level

Theorem (Gross-Vignéras, Serre: Conjecture 3.2.6?)

Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(V)$ be a continuous, odd, irreducible representation of the absolute Galois group over $\mathbb{Q}$ to a 2-dimensional $\overline{\mathbb{F}}_\ell$-vector space V. Let $n = \mathrm{N}(\rho)$ and $k = k(\rho)$, let $f \in S(n,k)_{\overline{\mathbb{F}}_\ell}$ be an eigenform such that $\rho_f \cong \rho$. Let p be a prime divisor of ℓn.

1. *If $f(T_p) \neq 0$, then there exists a stable line $D \subset V$ for the action of G_p, the decomposition subgroup at p, such that the inertia group at p acts trivially on V/D. Moreover, $f(T_p)$ is equal to the eigenvalue of Frob_p which acts on V/D.*
2. *If $f(T_p) = 0$, then there exists no stable line $D \subset V$ as in (1).*

Proposition (A.)

Assume setting $(*)$ and that ρ_f is irreducible and it does not arise from lower level. Let p be a prime dividing n such that $f(T_p) \neq 0$.
Then $\rho_f|_{G_p}$ is decomposable if and only if $\rho_f|_{I_p}$ is decomposable.

This proposition is proved using representation theory.

Proposition (A.)

Assume setting $(*)$ *and that* ρ_f *is irreducible and it does not arise from lower level. Let* p *be a prime dividing* n*, such that* $f(T_p) \neq 0$*. Then:*

(A) $\rho_f|_{I_p}$ *is decomposable if and only if* $\mathrm{N}_p(\rho_f) = \mathrm{N}_p(\bar{\epsilon})$*;*

(B) $\rho_f|_{I_p}$ *is indecomposable if and only if* $\mathrm{N}_p(\rho_f) = 1 + \mathrm{N}_p(\bar{\epsilon})$*.*

Sketch of the proof

The valuation of $\mathrm{N}(\rho_f)$ at p is given by:

$$\mathrm{N}_p(\rho_f) = \sum_{i \geq 0} \frac{1}{[G_{0,p} : G_{i,p}]} \dim(V/V^{G_{i,p}}) = \dim(V/V^{I_p}) + b(V),$$

where V is the two-dimensional $\overline{\mathbb{F}}_\ell$-vector space underlying the representation, $V^{G_{i,p}}$ is its subspace of invariants under $G_{i,p}$, and $b(V)$ is the wild part of the conductor.
Since $f(T_p) \neq 0$, the representation restricted to the decomposition group at p is reducible. Hence, after conjugation,

$$\rho_f|_{G_p} \cong \begin{pmatrix} \epsilon_1 \chi_\ell^{k-1} & * \\ 0 & \epsilon_2 \end{pmatrix}, \quad \rho_f|_{I_p} \cong \begin{pmatrix} \epsilon_1|_{I_p} & * \\ 0 & 1 \end{pmatrix} \cong \begin{pmatrix} \overline{\epsilon}|_{I_p} & * \\ 0 & 1 \end{pmatrix}$$

where ϵ_1 and ϵ_2 are characters of G_p with ϵ_2 unramified, χ_ℓ is the mod ℓ cyclotomic character and $*$ belongs to $\overline{\mathbb{F}}_\ell$.

Remark

If $\rho_f|_{I_p}$ is indecomposable then the image of inertia at p is of order divisible by ℓ and so the image cannot be exceptional, hence it is **big**.

Let n be a positive integer. Any Dirichlet character of conductor n can be decomposed into local characters, one for each prime divisor of n.

With no loss of generality, we reduce ourselves to study twists of modular Galois representations with Dirichlet characters with prime power conductor.

Question

What is the conductor of the twist?

Shimura gave an upper bound:

$$\boxed{\operatorname{lcm}(\operatorname{cond}(\chi)^2, n)}$$

where n is the level of the form and χ is the character used for twisting.

Proposition (A.)

Assume setting $(*)$*. Let* p *be a prime* **not** *dividing* $n\ell$*. Let* $\chi : (\mathbb{Z}/p^i\mathbb{Z})^* \to \overline{\mathbb{F}}_\ell^*$*, for* $i > 0$*, be a non-trivial character. Then*

$$\mathrm{N}_p(\rho_f \otimes \chi) = 2\mathrm{N}_p(\chi).$$

Proposition (A.)

Assume setting $(*)$ and that ρ_f is irreducible and it does not arise from lower level. Let p be a prime dividing n and suppose that $f(T_p) \neq 0$. Let $\chi : (\mathbb{Z}/p^i\mathbb{Z})^* \to \overline{\mathbb{F}}_\ell^*$, for $i > 0$, be a non-trivial character. Then

$$\mathrm{N}_p(\rho_f \otimes \chi) = \mathrm{N}_p(\chi\bar{\epsilon}) + \mathrm{N}_p(\chi).$$

It is also possible to know what is the system of eigenvalues associated to the twist:

Proposition (A.)

Assume setting $(*)$. Suppose that ρ_f is irreducible and that $\mathrm{N}(\rho_f) = n$. Let p be a prime dividing n and suppose that $f(T_p) \neq 0$. Let χ from $(\mathbb{Z}/p^i\mathbb{Z})^*$ to $\overline{\mathbb{F}}_\ell^*$, with $i > 0$, be a non-trivial character. Then

(A) *if $\rho_f|_{I_p}$ is decomposable then the representation $\rho_f \otimes \chi$ restricted to G_p, the decomposition group at p, admits a stable line with unramified quotient if and only if $\mathrm{N}_p(\rho_f \otimes \chi) = \mathrm{N}_p(\rho_f)$;*

(B) *if $\rho_f|_{I_p}$ is indecomposable then the representation $\rho_f \otimes \chi$ restricted to G_p does not admit any stable line with unramified quotient.*

Proposition (A.)

Assume setting $(*)$. Suppose that ρ_f is irreducible and that $\mathrm{N}(\rho_f) = n$. Let p be a prime dividing n and suppose that $f(T_p) = 0$. Then:

(A) *if $\rho_f|_{G_p}$ is reducible then there exists a mod ℓ modular form g of weight k and level at most np and a non-trivial character $\chi : (\mathbb{Z}/p^i\mathbb{Z})^* \to \overline{\mathbb{F}}_\ell^*$ with $i > 0$ such that $g(T_p) \neq 0$ and $\rho_g \cong \rho_f \otimes \chi$;*

(B) *if $\rho_f|_{G_p}$ is irreducible then for any non-trivial character $\chi : (\mathbb{Z}/p^i\mathbb{Z})^* \to \overline{\mathbb{F}}_\ell^*$ with $i > 0$ the representation $\rho_f \otimes \chi$ restricted to G_p does not admit any stable line with unramified quotient.*

Example 1
Let $n = 135 = 3^3 5$. Let ϵ be the Dirichlet character modulo 135 of conductor 5 mapping $56 \to 1$, $82 \to \zeta_{36}^{9}$.

$S(135, 3, \epsilon)^{\mathrm{new}}_{\mathbb{C}}$ two Galois orbits: the two Hecke eigenvalue fields are:
$\mathbb{Q}\left(x^{16} + 217x^{12} + 9264x^{8} + 59497x^{4} + 28561\right)$ and
$\mathbb{Q}\left(x^{16} + 286x^{12} + 16269x^{8} + 85684x^{4} + 62500\right)$.

In the first one, applying the reduction map for all prime ideals above 7, we obtain the following eigenvalue systems defined over
$\mathbb{F}_{7^2} \cong \mathbb{F}_7[x]/[x^2 + 6x + 4] \cong \mathbb{F}_7[\alpha]$:

p	$f_1(T_p)$	$f_2(T_p)$	$f_3(T_p)$
2	α	6α	$\alpha+6$
3	0	0	0
5	$5\alpha+4$	$2\alpha+3$	$5\alpha+5$
7	0	0	0
11	$\alpha+3$	$6\alpha+4$	$\alpha+3$
13	6α	6α	$\alpha+6$
17	6α	α	$6\alpha+1$
19	$6\alpha+4$	$6\alpha+4$	$\alpha+3$
23	$3\alpha+4$	$4\alpha+3$	3α

f_1,f_2 and f_3 are mod 7 modular forms of level 135 and weight 3.

It is easy to verify that the corresponding representations are of minimal level and weight.

Let us focus on f_1.

Since 5 divides the level and $f_1(T_5) \neq 0$:

$$\rho_{f_1}|_{G_5} \cong \begin{pmatrix} \epsilon_1 \chi_7^2 & * \\ 0 & \epsilon_2 \end{pmatrix} \cong \begin{pmatrix} \epsilon_2^{-1} \epsilon_{f_1} \chi_7^2 & 0 \\ 0 & \epsilon_2 \end{pmatrix}$$

The character of f_1 has conductor 5 hence the representation is reducible and decomposable, so $* = 0$. Moreover $\epsilon_2(5) = f_1(T_5)$.

If we twist by $\epsilon_{f_1}^{-1}$ then we have:

$$(\rho_{f_1} \otimes \epsilon_{f_1}^{-1})|_{G_5} \cong \begin{pmatrix} \epsilon_2^{-1} \chi_7^2 & 0 \\ 0 & \epsilon_2 \epsilon_{f_1}^{-1} \end{pmatrix}$$

hence we know the eigenvalue at 5. The computation on the conductor gives that the conductor is 135 and $(\epsilon_2^{-1}\chi_7^2)(5) = 4\epsilon_2^{-1}(5) = 5\alpha + 5$.

p	$f_1(T_p)$	$f_2(T_p)$	$f_3(T_p)$	$\rho_{f_1} \otimes \epsilon_{f_1}^{-1}$
2	α	6α	$\alpha+6$	$\alpha+6$
3	0	0	0	0
5	$5\alpha+4$	$2\alpha+3$	$5\alpha+5$	$5\alpha+5$
7	0	0	0	0
11	$\alpha+3$	$6\alpha+4$	$\alpha+3$	$\alpha+3$
13	6α	6α	$\alpha+6$	$\alpha+6$
17	6α	α	$6\alpha+1$	$6\alpha+1$
19	$6\alpha+4$	$6\alpha+4$	$\alpha+3$	$\alpha+3$
23	$3\alpha+4$	$4\alpha+3$	3α	3α
29	2	5	5	5
31	4	4	4	4
37	$\alpha+6$	$\alpha+6$	6α	6α
41	$5\alpha+1$	$2\alpha+6$	$5\alpha+1$	$5\alpha+1$
43	α	α	$6\alpha+1$	$6\alpha+1$
47	2α	5α	$2\alpha+5$	$2\alpha+5$

The level is also divisible by 3. But $f_1(T_3) = 0$.

The local representation at 3 is irreducible: in order to prove this we have to check all lover levels and possible twist.
In this case it is easy since the newforms space is empty in most cases.
The argument is similar for all levels, so we will just show what happens for level 15.

p	$f_1(T_p)$	$g_1(T_p)$	$g_2(T_p)$	$g_3(T_p)$	$g_4(T_p)$
2	α	$6\alpha + 6$	$\alpha + 5$	$4\alpha + 1$	$3\alpha + 5$
3	0	6α	$\alpha + 6$	α	$6\alpha + 1$
5	$5\alpha + 4$	$3\alpha + 5$	$4\alpha + 1$	$2\alpha + 1$	$5\alpha + 3$
7	0	$4\alpha + 5$	$3\alpha + 2$	2	2
11	$\alpha + 3$	$6\alpha + 1$	α	α	$6\alpha + 1$
13	6α	$4\alpha + 5$	$3\alpha + 2$	5	5

It is possible to show that the image of the Galois representation up to conjugation is

$$\rho_{f_1}(G_{\mathbb{Q}}) \cong < \alpha > \mathrm{SL}_2(\mathbb{F}_7) \subset \mathrm{GL}_2(\mathbb{F}_{7^2})$$

$(f(T_p))^2(\epsilon_f(p)p^2)^{-1} \mapsto 6, 0, 6, 4, 5, 4, 5, 3, 1, 1, 6, 4, 3$

Example 2

Let us consider $S(40, 2, \tau)^{\text{new}}_{\mathbb{C}}$ where τ is the quadratic character conductor 40 mapping $31 \to 1$,$21 \to -1$, $17 \to -1$.

As before, by reduction get mod 7 eigenvalue systems, for example f_i for $i = 1, 2, 3, 4$:

p	$f_1(T_p)$	$f_2(T_p)$	$f_3(T_p)$	$f_4(T_p)$
2	6	1	5	2
3	3	4	3	4
5	2	1	6	5
7	6	1	1	6
11	4	4	3	3
13	0	0	0	0
17	2	5	5	2
19	3	3	4	4
23	1	6	6	1

Let χ be the character over $\mathbb{F}_7$ of conductor 8 such that $\chi\epsilon_{f_1}$ has conductor 4. Since $f_1(T_2) \neq 0$, then we have that:

$$\mathrm{N}_2(\rho_f \otimes \chi) = \mathrm{N}_2(\chi\epsilon_{f_1}) + \mathrm{N}_2(\chi) = 2 + 3 = 5 \neq 6.$$

Hence, let us check the twist and the eigenvalue systems at level 160: $\mathrm{N}_2(\rho_f \otimes \chi) = 2^5 5 = 160$:

p	$g_1(T_p)$	$g_2(T_p)$	$g_3(T_p)$	$g_4(T_p)$	$\rho_{f_1} \otimes \chi$	$\rho_{f_2} \otimes \chi$	$\rho_{f_3} \otimes \chi$	$\rho_{f_4} \otimes \chi$
2	0	0	0	0	0	0	0	0
3	4	3	4	3	3	4	3	4
5	6	5	2	1	5	6	1	2
7	6	1	1	6	1	6	6	1
11	4	4	3	3	4	4	3	3
13	0	0	0	0	0	0	0	0
17	5	2	2	5	2	5	5	2
19	3	3	4	4	3	3	4	4
23	1	6	6	1	6	1	1	6

All the systems given by g_i are reduction of the same form.

Also in this case it is possible to prove that the image of the Galois representation is big: in this case, it is isomorphic to $GL_2(\mathbb{F}_7)$, up to conjugation.

Example 3
$S(7,3)^{\mathrm{new}}_{\mathbb{C}}$, $S(49,3)^{\mathrm{new}}_{\mathbb{C}}$

The Hecke eigenvalue fields in newformspace at level 49 are given by:

- $\mathbb{Q}\left(x^2 - 3x + 9\right)$
- $\mathbb{Q}\left(x^4 - 4x^3 + 22x^2 - 44x + 167\right)$
- $\mathbb{Q}\left(x^8 + 4x^7 - 6x^6 - 96x^5 + 225x^4 + 1336x^3 - 514x^2 - 5948x + 7399\right)$
- Number Field with defining polynomial $x^{54} + 5x^{53} + 43x^{52} + 169x^{51} + \cdots + 248413945171829320x^{16} + \cdots + 348605788594202619x^8 + \cdots + 10909749546081$
- Number Field with defining polynomial $x^{96} + 13x^{95} + 69x^{94} + 174x^{93} + 16x^{92} - 1672x^{91} + \ldots$

while in level 7 ... the Hecke eigenvalue field is $\mathbb{Q}$.

Let $\mathbb{F}_{5^2} \cong \mathbb{F}_5[x]/[x^2 + 2x + 4] \cong \mathbb{F}_5[\beta]$.

Let χ be Dirichlet character over $\mathbb{F}_{5^2}$ modulo 49 of conductor 7 mapping $3 \to 3\beta$.

p	$f(T_p)$	$g(T_p)$	$\rho_f \otimes \chi$
2	2	β	β
3	0	0	0
5	0	0	0
7	3	0	0
11	4	$3\beta+1$	$3\beta+1$
13	0	0	0
17	0	0	0
19	0	0	0
23	3	4β	4β
29	1	1	1

In this case we have that $\mathbb{P}\rho_f \cong A_4$ and $\rho_f(G_{\mathbb{Q}}) \cong \mathbb{F}_5^* \pi^{-1}(A_4)$, where A_4 is the alternating group of 4 elements.

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Thanks!