

THE INVERSE GALOIS PROBLEM AND UNIFORM REALIZATIONS

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Université Blaise Pascal, 15th March 2016

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THE INVERSE GALOIS PROBLEM

Let G be a finite group. Does there exist a Galois extension K/\mathbb{Q} such that $\text{Gal}(K/\mathbb{Q}) \cong G$?

For example, let G be S_n , the symmetric group of n letters. Then G is a Galois group over \mathbb{Q} . Moreover, for all positive integer n we can realize G as the Galois group of the splitting field $x^n - x - 1$.

Galois representations may answer the inverse Galois problem for finite linear groups.

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Let $\overline{\mathbb{Q}}$ be an algebraic closure of \mathbb{Q} and let $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Let A be a principally polarized abelian variety over \mathbb{Q} of dimension d .

Let ℓ be a prime and $A[\ell]$ the ℓ -torsion subgroup:

$$A[\ell] := \{P \in A(\overline{\mathbb{Q}}) \mid [\ell]P = 0\} \cong (\mathbb{Z}/\ell\mathbb{Z})^{2d}.$$

$A[\ell]$ is a $2d$ -dimensional \mathbb{F}_{ℓ} -vector space, as well as a $G_{\mathbb{Q}}$ -module.

The polarization induces a symplectic pairing, the mod ℓ Weil pairing on $A[\ell]$, which is a bilinear, alternating, non-degenerate pairing:

$$\langle \cdot, \cdot \rangle : A[\ell] \times A[\ell] \rightarrow \mu_\ell$$

that is Galois invariant: $\forall \sigma \in G_{\mathbb{Q}}, \forall v, w \in A[\ell]$

$$\langle \sigma v, \sigma w \rangle = \chi(\sigma) \langle v, w \rangle,$$

where $\chi : G_{\mathbb{Q}} \rightarrow \mathbb{F}_\ell^\times$ is the mod ℓ cyclotomic character.

$(A[\ell], \langle \cdot, \cdot \rangle)$ is a symplectic \mathbb{F}_ℓ -vector space of dimension $2d$. This gives a representation

$$\bar{\rho}_{A,\ell} : G_{\mathbb{Q}} \rightarrow \mathrm{GSp}(A[\ell], \langle \cdot, \cdot \rangle) \cong \mathrm{GSp}_{2d}(\mathbb{F}_\ell).$$

THEOREM (SERRE)

Let A be a principally polarized abelian variety of dimension d , defined over \mathbb{Q} . Assume that $d = 2, 6$ or d is odd and, furthermore, assume that $\text{End}_{\overline{\mathbb{Q}}}(A) = \mathbb{Z}$. Then there exists a bound B_A such that for all primes $\ell > B_A$ the representation $\bar{\rho}_{A,\ell}$ is surjective.

The conclusion of the theorem is known to be false for general d (counterexample by Mumford for $d = 4$).

OPEN QUESTION

Given d as in the theorem, is there a uniform bound B_d depending only on d , such that for all principally polarized abelian varieties A over \mathbb{Q} of dimension d with $\text{End}_{\overline{\mathbb{Q}}}(A) = \mathbb{Z}$, and all $\ell > B_d$, the representation $\bar{\rho}_{A,\ell}$ is surjective?

For elliptic curves an affirmative answer is expected, and this is known as Serre's Uniformity Question.

Much easier for semistable elliptic curves:

THEOREM (SERRE)

Let E/\mathbb{Q} be a semistable elliptic curve, and $\ell \geq 11$ be a prime. Then $\bar{\rho}_{E,\ell}$ is surjective.

BACK TO THE INVERSE GALOIS PROBLEM

The Galois representation attached to the ℓ -torsion of the elliptic curve $y^2 + y = x^3 - x$ is surjective for all prime ℓ . This gives a realization $\mathrm{GL}_2(\mathbb{F}_\ell)$ as Galois group for all ℓ .

Let C be the genus 2 hyperelliptic curve given by $y^2 = x^5 - x + 1$ and let J denotes its Jacobian. Dieulefait proved that $\bar{\rho}_{J,\ell}$ is surjective for all odd prime ℓ . This gives a realization $\mathrm{GSp}_4(\mathbb{F}_\ell)$ as Galois group for all odd ℓ .

$\mathrm{GSp}_6(\mathbb{F}_\ell)$

What about genus 3 curves?

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THEOREM (A., LEMOS AND SIKSEK)

Let A be a semistable principally polarized abelian variety of dimension $d \geq 1$ over \mathbb{Q} and let $\ell \geq \max(5, d + 2)$ be prime.

Suppose the image of $\bar{\rho}_{A,\ell} : G_{\mathbb{Q}} \rightarrow \mathrm{GSp}_{2d}(\mathbb{F}_{\ell})$ contains a transvection. Then $\bar{\rho}_{A,\ell}$ is either reducible or surjective.

THEOREM (A., LEMOS AND SIKSEK)

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*Suppose the image of $\bar{\rho}_{A,\ell} : G_{\mathbb{Q}} \rightarrow \mathrm{GSp}_{2d}(\mathbb{F}_{\ell})$ contains a **transvection**.
Then $\bar{\rho}_{A,\ell}$ is either reducible or surjective.*

TRANSVECTION

DEFINITION

Let $(V, \langle \ , \ \rangle)$ be a finite-dimensional symplectic vector space over \mathbb{F}_ℓ . A **transvection** is an element $T \in \mathrm{GSp}(V, \langle \ , \ \rangle)$ which fixes a hyperplane $H \subset V$.

Therefore, a transvection is a unipotent element $\sigma \in \mathrm{GSp}(V, \langle \ , \ \rangle)$ such that $\sigma - I$ has rank 1.

WHEN DOES $\bar{\rho}_{A,\ell}(G_{\mathbb{Q}})$ CONTAIN A TRANSVECTION?

Let $q \neq \ell$ be a prime and suppose that the following two conditions are satisfied:

- the special fibre of the Néron model for A at q has toric dimension 1;
- $\ell \nmid \#\Phi_q$, where Φ_q is the group of connected components of the special fibre of the Néron model at q .

Then the image of $\bar{\rho}_{A,\ell}$ contains a transvection (Hall).

WHEN DOES $\bar{\rho}_{A,\ell}(G_{\mathbb{Q}})$ CONTAIN A TRANSVECTION?

Let C/\mathbb{Q} be a hyperelliptic curve of genus d :

$$C : y^2 = f(x)$$

where $f \in \mathbb{Z}[x]$ is a squarefree polynomial.

Let p be an odd prime not dividing the leading coefficient of f such that f modulo p has one root in $\bar{\mathbb{F}}_p$ having multiplicity precisely 2, with all other roots simple.

Then the Néron model of the Jacobian at p has toric dimension 1 (Hall).

INGREDIENTS OF THE PROOF OF THE MAIN THEOREM

In the proof of this theorem we rely on:

- the classification due to Arias-de-Reyna, Dieulefait and Wiese of subgroups of $\mathrm{GSp}_{2d}(\mathbb{F}_\ell)$ containing a transvection;
- results of Raynaud on the image of the inertia subgroup.

CLASSIFICATION OF SUBGROUPS OF $\mathrm{GSp}_{2d}(\mathbb{F}_\ell)$ WITH A TRANSVECTION

THEOREM (ARIAS-DE-REYNA, DIEULEFAIT AND WIESE)

Let $\ell \geq 5$ be a prime and let V a symplectic \mathbb{F}_ℓ -vector space of dimension $2d$. Any subgroup G of $\mathrm{GSp}(V)$ which contains a transvection satisfies one of the following:

- (I) There is a non-trivial proper G -stable subspace $W \subset V$.
- (II) There are non-singular symplectic subspaces $V_i \subset V$ with $i = 1, \dots, h$, of dimension $2m < 2d$ and a homomorphism $\phi : G \rightarrow S_h$ such that $V = \bigoplus_{i=1}^h V_i$ and $\sigma(V_i) = V_{\phi(\sigma)(i)}$ for $\sigma \in G$ and $1 \leq i \leq h$. Moreover, $\phi(G)$ is a transitive subgroup of S_h .
- (III) $\mathrm{Sp}(V) \subseteq G$.

We apply this to $G = \bar{\rho}_{A,\ell}(G_{\mathbb{Q}})$ where A and ℓ are as in the main theorem. If $\mathrm{Sp}_{2d}(\mathbb{F}_\ell) \subseteq G$ then $G = \mathrm{GSp}_{2d}(\mathbb{F}_\ell)$, since the mod ℓ cyclotomic character is surjective.

INERTIA AND A THEOREM OF RAYNAUD

THEOREM (RAYNAUD)

Let A be an abelian variety over \mathbb{Q} . Let ℓ be a prime of semistable reduction for A . Regard $A[\ell]$ as an I_ℓ -module and let V be a Jordan-Hölder factor of dimension n over \mathbb{F}_ℓ . Let $\psi_n : I_\ell \rightarrow \mathbb{F}_{\ell^n}^\times$ be a fundamental character of level n . Then V has the structure of a 1-dimensional \mathbb{F}_{ℓ^n} -vector space and the action of I_ℓ on it is given by a character $\varpi : I_\ell \rightarrow \mathbb{F}_{\ell^n}^\times$, where $\varpi = \psi_n^{\sum_{i=0}^{n-1} a_i \ell^i}$ with $a_i = 0$ or 1 .

IDEA OF THE PROOF

Denote $\bar{\rho} = \bar{\rho}_{A,\ell}$. Let $G = \bar{\rho}(G_{\mathbb{Q}}) \subseteq \mathrm{GSp}_{2d}(\mathbb{F}_{\ell})$.

Since G contains a transvection, it is sufficient to show that the induced case does not arise.

Suppose otherwise. Write $V = \bigoplus_{i=1}^h V_i$ where V_i are non-singular symplectic subspaces of dimension $2m < 2d$. Then there is some $\phi : G \rightarrow S_h$ with transitive image such that $\sigma(V_i) = V_{\phi(\sigma)(i)}$. Let

$$\begin{array}{ccccc}
 & & \pi & & \\
 & \curvearrowright & & \curvearrowright & \\
 G_{\mathbb{Q}} & \xrightarrow{\bar{\rho}} & G & \xrightarrow{\phi} & S_h
 \end{array}$$

IDEA OF THE PROOF

$$\begin{array}{ccccc}
 & & \pi & & \\
 & \curvearrowright & & \curvearrowleft & \\
 G_{\mathbb{Q}} & \xrightarrow{\bar{\rho}} & G & \xrightarrow{\phi} & S_h
 \end{array}$$

Let $H = \ker(\pi)$. Then $H = G_K$ for some number field K/\mathbb{Q} . Moreover, $\bar{\rho}|_{G_K}$ is reducible as the V_i are stable under the action of G_K .

In the proof we show that for $\ell \geq \max(5, d + 2)$ the extension K/\mathbb{Q} is unramified at the finite places, and thus K has discriminant ± 1
 $\Rightarrow K = \mathbb{Q} \Rightarrow \pi$ is trivial \Rightarrow contradiction

IDEA OF THE PROOF

$$\begin{array}{ccccc}
 & & \pi & & \\
 & \curvearrowright & & \curvearrowright & \\
 G_{\mathbb{Q}} & \xrightarrow{\bar{\rho}} & G & \xrightarrow{\phi} & S_h
 \end{array}$$

Let $p \neq \ell$ be a prime. As A is semistable, I_p acts unipotently on V , so $\bar{\rho}(\sigma)$ has ℓ -power order for $\sigma \in I_p$.

The order of $\bar{\rho}(\sigma)$ is divisible by the order of $\pi(\sigma)$ which divides $h!$

As $h = 2d/2m \leq d < \ell \Rightarrow \pi(\sigma) = 1 \Rightarrow K/\mathbb{Q}$ is unramified at p .

The proof for $p = \ell$ is more involved. The bound on ℓ is obtained considering the image of inertia subgroup and applying Raynaud's result.

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We now let A/\mathbb{Q} be a **principally polarized abelian threefold**.

ASSUMPTIONS

- (A) A is semistable;
- (B) $\ell \geq 5$;
- (C) there is a prime q such that the special fibre of the Néron model for A at q has toric dimension 1.
- (D) ℓ does not divide $\gcd(\{q \cdot \#\Phi_q : q \in S\})$, where S is the set of primes q satisfying (C) and Φ_q is the group of connected components of the special fibre of the Néron model of A at q .

Under these assumptions the image of $\bar{\rho}_{A,\ell}$ contains a transvection. Then $\bar{\rho}_{A,\ell}$ is either reducible or surjective.

"ALGORITHM"

Practical method which should, in most cases, produce a small integer B (depending on A) such that for $\ell \nmid B$, the representation $\bar{\rho}_{A,\ell}$ is irreducible and, hence, surjective.

We will apply this procedure to the Jacobian of the hyperelliptic curve:

$$C : y^2 + (x^4 + x^3 + x + 1)y = x^6 + x^5.$$

DETERMINANTS OF JORDAN–HÖLDER FACTORS

Let $\chi : G_{\mathbb{Q}} \rightarrow \mathbb{F}_{\ell}^{\times}$ denote the mod ℓ cyclotomic character.

We will study the Jordan–Hölder factors W of the $G_{\mathbb{Q}}$ -module $A[\ell]$.
By the determinant of such a W we mean the determinant of the induced representation $G_{\mathbb{Q}} \rightarrow \mathrm{GL}(W)$.

LEMMA

Any Jordan–Hölder factor W of the $G_{\mathbb{Q}}$ -module $A[\ell]$ has determinant χ^r for some $0 \leq r \leq \dim(W)$.

WEIL POLYNOMIALS

From a prime $p \neq \ell$ of good reduction for A , we will denote by

$$P_p(x) = x^6 + \alpha_p x^5 + \beta_p x^4 + \gamma_p x^3 + p\beta_p x^2 + p^2\alpha_p + p^3 \in \mathbb{Z}[x]$$

the characteristic polynomial of Frobenius $\sigma_p \in G_{\mathbb{Q}}$ at p acting on the Tate module $T_{\ell}(A)$ (also known as the **Weil polynomial** of $A \bmod p$). The polynomial P_p is independent of ℓ .

Its roots in $\overline{\mathbb{F}}_{\ell}$ have the form $u, v, w, p/u, p/v, p/w$.

1-DIMENSIONAL JORDAN–HÖLDER FACTORS

Let T be a non-empty set of primes of good reduction for A . Let

$$B_1(T) = \gcd(\{p \cdot \#A(\mathbb{F}_p) : p \in T\}).$$

LEMMA

Suppose $\ell \nmid B_1(T)$. The $G_{\mathbb{Q}}$ -module $A[\ell]$ does not have any 1-dimensional or 5-dimensional Jordan–Hölder factors.

2-DIMENSIONAL JORDAN–HÖLDER FACTORS

LEMMA

Suppose the $G_{\mathbb{Q}}$ -module $A[\ell]$ does not have any 1-dimensional Jordan–Hölder factors, but has either a 2-dimensional or 4-dimensional irreducible subspace U . Then $A[\ell]$ has a 2-dimensional Jordan–Hölder factor W with determinant χ .

Let N be the conductor of A . Let W be a 2-dimensional Jordan–Hölder factor of $A[\ell]$ with determinant χ . The representation

$$\tau : G_{\mathbb{Q}} \rightarrow \mathrm{GL}(W) \cong \mathrm{GL}_2(\mathbb{F}_{\ell})$$

is odd (as the determinant is χ), irreducible (as W is a Jordan–Hölder factor) and 2-dimensional. By Serre's modularity conjecture (Khare, Wintenberger, Dieulefait, Kisin Theorem), this representation is **modular**:

$$\tau \cong \bar{\rho}_{f,\ell}$$

it is equivalent to the mod ℓ representation attached to a newform f of level $M \mid N$ and weight 2.

Let \mathcal{O}_f be the ring of integers of the number field generated by the Hecke eigenvalues of f . Then there is a prime $\lambda \mid \ell$ of \mathcal{O}_f such that for all primes $p \nmid \ell N$,

$$\mathrm{Tr}(\tau(\sigma_p)) \equiv c_p(f) \pmod{\lambda}$$

where $\sigma_p \in G_{\mathbb{Q}}$ is a Frobenius element at p and $c_p(f)$ is the p -th Hecke eigenvalue of f .

As W is a Jordan–Hölder factor of $A[\ell]$ we see that $x^2 - c_p(f)x + p$ is a factor modulo λ of P_p .

Now let $H_{M,p}$ be the p -th Hecke polynomial for the new subspace $S_2^{\text{new}}(M)$ of cusp forms of weight 2 and level M . This has the form

$$H_{M,p} = \prod (x - c_p(g)),$$

where g runs through the newforms of weight 2 and level M . Write

$$H'_{M,p}(x) = x^d H_{M,p}(x + p/x) \in \mathbb{Z}[x],$$

where $d = \deg(H_{M,p}) = \dim(S_2^{\text{new}}(M))$.

It follows that $x^2 - c_p(f)x + p$ divides $H'_{M,p}$.

Let

$$R(M, p) = \text{Res}(P_p, H'_{M,p}) \in \mathbb{Z},$$

where Res denotes resultant. If $R(M, p) \neq 0$ then we have a bound on ℓ .

The integers $R(M, p)$ can be very large. Given a non-empty set T of rational primes p of good reduction for A , let

$$R(M, T) = \gcd(\{p \cdot R(M, p) : p \in T\}).$$

In practice, we have found that for a suitable choice of T , the value $R(M, T)$ is fairly small.

Let

$$B'_2(T) = \text{lcm}(R(M, T))$$

where M runs through the divisors of N such that $\dim(S_2^{\text{new}}(M)) \neq 0$,
and let

$$B_2(T) = \text{lcm}(B_1(T), B'_2(T))$$

where $B_1(T)$ is given as before.

LEMMA

Let T be a non-empty set of rational primes of good reduction for A , and suppose $\ell \nmid B_2(T)$. Then $A[\ell]$ does not have 1-dimensional Jordan-Hölder factors, and does not have irreducible 2- or 4-dimensional subspaces.

We fail to bound ℓ in the above lemma if $R(M, p) = 0$ for all primes p of good reduction.

Here are two situations where this can happen:

- Suppose $A \cong_{\mathbb{Q}} E \times A'$ where E is an elliptic curve and A' an abelian surface. Let $M \mid N$ be the conductor of the elliptic curve, and f to be the newform associated to E by modularity, then $x^2 - c_p(f)x + p$ is a factor of $P_p(x) \Rightarrow R(M, p) = 0$ for all $p \nmid N$.
- Suppose A is of GL_2 -type. Let f be the corresponding eigenform, then again $x^2 - c_p(f)x + p$ is a factor of $P_p(x)$ in $\mathcal{O}_f[x] \Rightarrow R(M, p) = 0$ for all $p \nmid N$.

Note that in both these situations $\text{End}_{\overline{\mathbb{Q}}}(A) \neq \mathbb{Z}$.

We expect, but are unable to prove, that if $\text{End}_{\overline{\mathbb{Q}}}(A) = \mathbb{Z}$ then there will be primes p such that $R(M, p) \neq 0$.

3-DIMENSIONAL JORDAN–HÖLDER FACTORS

LEMMA

Suppose $A[\ell]$ has Jordan–Hölder filtration $0 \subset U \subset A[\ell]$ where both U and $A[\ell]/U$ are irreducible and 3-dimensional. Moreover, let u_1, u_2, u_3 be a basis for U , and let

$$G_{\mathbb{Q}} \rightarrow \mathrm{GL}_3(\mathbb{F}_{\ell}), \quad \sigma \mapsto M(\sigma)$$

give the action of $G_{\mathbb{Q}}$ on U with respect to this basis. Then we can extend u_1, u_2, u_3 to a symplectic basis $u_1, u_2, u_3, w_1, w_2, w_3$ for $A[\ell]$ so that the action of $G_{\mathbb{Q}}$ on $A[\ell]$ with respect to this basis is given by

$$G_{\mathbb{Q}} \rightarrow \mathrm{GSp}_6(\mathbb{F}_{\ell}), \quad \sigma \mapsto \left(\begin{array}{c|c} M(\sigma) & * \\ \hline \mathbf{0} & \chi(\sigma)(M(\sigma)^t)^{-1} \end{array} \right).$$

$\det(U) = \chi^r$ and $\det(A[\ell]/U) = \chi^s$ where $0 \leq r, s \leq 3$ with $r + s = 3$.

LEMMA

Let p be a prime of good reduction for A . For ease write α , β and γ for the coefficients α_p , β_p , γ_p in the equation of the Weil polynomial. Suppose $p + 1 \neq \alpha$. Let

$$\delta = \frac{-p^2\alpha + p^2 + p\alpha^2 - p\alpha - p\beta + p - \beta + \gamma}{(p-1)(p+1-\alpha)} \in \mathbb{Q}, \quad \epsilon = \delta + \alpha \in \mathbb{Q}.$$

Let $g(x) = (x^3 + \epsilon x^2 + \delta x - p)(x^3 - \delta x^2 - p\epsilon x - p^2) \in \mathbb{Q}[x]$. Write k for the greatest common divisor of the numerators of the coefficients in $P_p - g$. Let

$$K_p = p(p-1)(p+1-\alpha)k.$$

Then $K_p \neq 0$. Moreover, if $\ell \nmid K_p$ then $A[\ell]$ does not have a Jordan-Hölder filtration as in the previous Lemma with $\det(U) = \chi$ or χ^2 .

LEMMA

Let p be a prime of good reduction for A . Write α , β and γ for the coefficients α_p , β_p , γ_p in the equation of the Weil polynomial. Suppose $p^3 + 1 \neq p\alpha$. Let $\epsilon' = p\delta' + \alpha \in \mathbb{Q}$ where

$$\delta' = \frac{-p^5\alpha + p^4 + p^3\alpha^2 - p^3\beta - p^2\alpha + p\gamma + p - \beta}{(p^3 - 1)(p^3 + 1 - p\alpha)} \in \mathbb{Q}.$$

Let $g'(x) = (x^3 + \epsilon'x^2 + \delta'x - 1)(x^3 - p\delta'x^2 - p^2\epsilon'x - p^3) \in \mathbb{Q}[x]$. Write k' for the greatest common divisor of the numerators of the coefficients in $P_p - g'$. Let

$$K'_p = p(p^3 - 1)(p^3 + 1 - p\alpha)k'.$$

Then $K'_p \neq 0$. Moreover, if $\ell \nmid K'_p$ then $A[\ell]$ does not have a Jordan-Hölder filtration as in the above Lemma with $\det(U) = 1$ or χ^3 .

SUMMARY

The following theorem summarizes all the lemmas:

THEOREM (A., LEMOS AND SIKSEK)

Let A and ℓ satisfy conditions (A)–(D). Let T be a non-empty set of primes of good reduction for A . Let

$$B_3(T) = \gcd(\{K_p : p \in T\}), \quad B_4(T) = \gcd(\{K'_p : p \in T\}),$$

where K_p and K'_p are defined in the last two Lemmas. Let

$$B(T) = \text{lcm}(B_2(T), B_3(T), B_4(T)).$$

If $\ell \nmid B(T)$ then $\bar{\rho}_{A,\ell}$ is surjective.

EXAMPLE

THEOREM (A., LEMOS AND SIKSEK)

Let C/\mathbb{Q} be the following genus 3 hyperelliptic curve,

$$C : y^2 + (x^4 + x^3 + x + 1)y = x^6 + x^5.$$

and write J for its Jacobian. Then

$$\bar{\rho}_{J,\ell}(G_{\mathbb{Q}}) = \mathrm{GSp}_6(\mathbb{F}_{\ell})$$

for all odd prime ℓ .

PROOF.

For $\ell \geq 5$ we apply the algorithm. For $\ell = 3$, we prove the result by direct computations. □

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FUTURE RESEARCH

- Generalization over **number fields**: obstruction coming from the Weil pairing, e.g.

$$E : y^2 + \left(\frac{\sqrt{101} + 1}{2}\right)y = x^3 + x^2 - 2x - 7 \quad \text{over } \mathbb{Q}(\sqrt{101})$$

$$\bar{\rho}_{E,\ell}(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\sqrt{101}))) \cong \text{GL}_2(\mathbb{F}_\ell) \quad \forall \text{ prime } \ell \neq 101$$

$$\rho_{E,101}(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\sqrt{101}))) \subseteq D \cdot \text{SL}_2(\mathbb{F}_{101})$$

where D is the set of invertible squares in \mathbb{F}_{101} .

- Generalization to **higher genus**.
- Generalization to **parametric families**, e.g. $y^2 = x^n - x + 1$.

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Thanks!