Extended formulations for Packing and Partitioning Orbitopes

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Outline

- What Packing and Partitioning Orbitopes are, and what for?
- P & P orbitopes in the original and in extended spaces.
- New proof of the complete description in the original space.
Packing and partitioning Orbitopes

Consider all $0-1$ matrices with $p$ rows and $q$ columns that have

1. exactly one non-zero entry per row;
2. columns in lexicographical non-increasing order;

Their convex hull is the partitioning orbitope $\mathbb{O}^-_{p,q}$.
Packing and partitioning Orbitopes

Consider all $0 - 1$ matrices with $p$ rows and $q$ columns that have

1. *at most* one non-zero entry per row;
2. columns in lexicographical non-increasing order;

Their convex hull is the *packing orbitope* $O_{p,q}^\leq$. 
Breaking symmetry in Graph Coloring

\begin{align*}
\text{min} & \quad \sum_{j=1}^{\lvert S \rvert} y_j \\
\text{s.t.} & \quad x_{ij} + x_{kj} \leq y_j, \quad \{i, k\} \in E, \; j \in S \\
& \quad \sum_{j=1}^{\lvert S \rvert} x_{ij} = 1, \quad i \in V \\
& \quad x_{ij} \in \{0, 1\}, \quad i \in V, j \in S \\
& \quad y_j \in \{0, 1\}, \quad j \in S
\end{align*}
Breaking symmetry using Orbitopes

- Any column-permutation of \( \begin{bmatrix} y^T \\ X \end{bmatrix} \) leads to an "equivalent solution"
- Thus, the solution space is unnecessarily large
- Idea: select one representative from each set of equivalent solutions.
- \( \text{IP} \cap O^\equiv_{p,q} \) as a symmetry-free formulation
- Re-use \( O^\equiv_{p,q} \) for different problems
- Change the number of 1s per row, "new" orbitopes!
Some facts on $P\bar{E}P$ Orbitopes

Introduced by Kaibel and Pfetsch [Math.Progr.08], who gave:

- an $O(p^2q)$ algorithm for optimizing a linear function over packing and partitioning orbitopes

- a complete ad irredudant description in the original space of $O^\equiv_{p,q}$ (resp. $O^{\leq}_{p,q}$) (SCI THR.):
  - $x \geq 0$
  - $x(row_i) = 1$ (resp. $x(row_i) \leq 1$)
  - exponentially many shifted column inequalities
Extended formulations

\[ P = \{ x \in \mathbb{R}^n | Ax \leq b \} \]
\[ Q = \{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m | Bx + Cy \leq d \} \]
\[ \text{Proj}_x(Q) = P \]
Lifting partitioning orbitopes (1)

Main idea:

Associate each integral \( x \in O_{p,q}^= \) with an \( s - t \) path on a digraph

Kipp Martin, Rardin, Campbell [Oper.Res.90]

\[ j(i) = \text{maximum column with a non-zero entry, up to row } i \]
Lifting partitioning orbitopes (1)

Main idea:
Associate each integral $x \in O_{p,q}$ with an $s - t$ path on a digraph
Kipp Martin, Rardin, Campbell [Oper.Res.90]

$j(i) =$ maximum column with
a non-zero entry, up to row $i$

$$j(i + 1) = j(i) \quad \text{or} \quad j(i + 1) = j(i) + 1$$
Lifting partitioning orbitopes (1)

Main idea:
Associate each integral $x \in \mathbb{O}_{p,q}^\equiv$ with an $s-t$ path on a digraph
Kipp Martin, Rardin, Campbell [Oper.Res.90]

$j(i) =$maximum column with
a non-zero entry, up to row $i$

$j(i+1) = j(i)$ or $j(i+1) = j(i) + 1$

A single path in general
corresponds to more vertices
of $\mathbb{O}_{p,q}^\equiv$
**Lifting partitioning orbitopes (2)**

\[
\max cx \text{ s.t. } x \in O_{p,q}^= \\
\]

\[
c_{(i,j)\downarrow} = c_{i+1,j+1} \\
c_{(i,j)\uparrow} = \max_{\ell \leq j} c_{i+1,\ell} \\
\]

**Thr. (F. & Kaibel 08):** The problem \( \max cx \text{ s.t. } x \in O_{p,q}^= \) can be solved in time \( O(pq) \).
The extended formulation

\[ P_{p,q} = \{(x, y)\mid y \in F_{p,q}, x(\text{row}_i) = 1, y(i-1,j-1) \leq x_{i,j}, \sum_{m=j}^{i} x_{i,m} \leq y(i-1,j-1) + \sum_{m=j}^{i-1} (y(i-1,m) + y(i-1,m+1)) \} \]

Thr. (F. & Kaibel 08): \( P_{p,q} \) is an extended formulation for \( O_{p,q}^= \).
Integrality of $P$ – sketch of the proof

Suppose you want to solve \( \max cx \text{ s.t. } x \in O_{p,q}^- \)

Define a new cost vector \( c^* \)

**Claim1**
\[ < c^*, (0, y) > \geq < c, (x, y) > \quad \forall (x, y) \in P_{p,q} \]

**Claim2**
For each integral \( y \in F_{p,q} \) exists integral \( x \) with \( (x, y) \in P_{p,q} \) and \[ < c, (x, y) > = < c^*, (0, y) >. \]
Integrality of $\mathcal{P}$ – sketch of the proof

Thus, for each $(\bar{x}, \bar{y}) \in P_{p,q}$:

\[ < c, (\bar{x}, \bar{y}) > \leq < c^*, (0, \bar{y}) > \leq \max_{y \in F_{p,q}} < c^*, (0, y) > \]

\[ = < c^*, (0, \tilde{y}) > \quad \text{for some integral } \tilde{y} \]

\[ = < c, (\tilde{x}, \tilde{y}) > \quad \text{for } \tilde{x} \text{ integral, } (\tilde{x}, \tilde{y}) \in P_{p,q} \]
More on the Extended Formulation

- Once you prove integrality, projecting is easy
- after suitable transformations, we end up with a "very compact" formulation with less than $2pq$ variables, $4pq$ constraints and $10pq$ total nonzero elements.
- (almost) identical results hold for $O_{p,q}^{\leq}$
  (actually, all the work is done for $O_{p,q}^{\leq}$)
Re-proving the SCI - theorem

The SCI-theorem is not necessary in our proofs

Use the extended formulation to obtain a new proof of the complete description in the original space. Why?

- Find a shorter proof
- Get new insight on the problem

Let $Q_{p,q}$ be the SCI-polytope. Since we already proved $\text{Proj}_x(P_{p,q}) = O_{p,q}^\perp$, we are left to prove

- $O_{p,q}^\perp \subseteq Q_{p,q}$;
- $Q_{p,q} \subseteq \text{Proj}_x(P_{p,q})$
Re-proving the SCI-theorem (1)

(SCI THR.): $O_{p,q}^-$ is completely described by:

- $x \geq 0$
- $x(row_i) = 1$
- exponentially many shifted column inequalities

New proof:

1. Start from a point $x$ s.t. $x(row_i) = 1$ for each $i$ and $x \geq 0$
2. consider a network on digraph $D$ with
   - capacity $+\infty$ on vertical arcs
   - capacity $x_{i,j}$ on the diagonal arc entering node $(i, j)$
3. Construct the rightmost flow $y = \Pi(x)$
(4) We shall prove that \((x, y = \Pi(x)) \in P_{p,q}\)
if the SCI are valid

(5) \(y_{(i-1,j-1)} \leq x_{i,j}\) come for free

(6) while trying to prove UB inequalities, SCI pops up!
Equivalent flows

\[ \geq 0 \]

\[ \geq 0 \]
Re-proving the SCI-theorem (3)

(7) Given \((i, j)\), build a backward leftmost flow

(8) SCIs imply UB on the \(x\) variables
Re-proving the SCI-theorem (3)

(7) Given \((i, j)\), build a backward leftmost flow

(8) SCIs imply UB on the \(x\) variables

\[ \geq 0 \]

Shifted column inequality!
Summary and conclusions

Extended formulations for Packing and Partitioning orbitopes...

- (very) compactly describe the polytopes;
- let us optimize faster;
- give more insight;
- shorten proofs;

Thank you!

Paper available at:

http://www.math.uni-magdeburg.de/~kaibel/