How hard is it to find extreme Nash equilibria in network congestion games?

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1. Problem Formulation
2. Preliminary Results
3. Complexity Results for Worst Nash Equilibria
4. Complexity Results for Best Nash Equilibria
The model

- A directed graph $G(V, E)$ with multiple edges
- A source $s$ and a sink $t$
- Non-decreasing latency functions $\ell_e : \mathbb{N}_0 \rightarrow \mathbb{R}_0^+$
- $N$ users, each routing the same amount of unsplittable flow
- Strategy set for all users: $\mathcal{P}$ — set of all simple $s$-$t$-paths
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```
x
2x
1.5x
```
The model

A flow is a function $f : \mathcal{P} \to \mathbb{N}_0$. The latency on a path $P \in \mathcal{P}$ is the sum of the latencies on its edges, i.e.,

$$\ell_P(f) := \sum_{e \in P} \ell_e \left( \sum_{P \in \mathcal{P} : e \in P} f_P \right)$$

Given a flow $f$ the social cost are given by

$$C_{\text{max}}(f) := \max_{P \in \mathcal{P} : f_P > 0} \ell_P(f).$$

\[ C_{\text{max}}(f) = \max\{1 + 3, 2 + 3, 1.5\} = 5 \]
Definition (Nash Equilibrium)

A flow $f$ is a Nash equilibrium, iff for all paths $P_1$, $P_2$ with $f_{P_1} > 0$ we have

$$\ell_{P_1}(f) \leq \ell_{P_2}(\tilde{f})$$

with

$$\tilde{f}_P = \begin{cases} 
    f_P - 1 & \text{if } P = P_1 \\
    f_P + 1 & \text{if } P = P_2 \\
    f_P & \text{otherwise}
\end{cases}$$

\[
\begin{array}{c}
\text{s} \quad \text{u} \quad \text{t} \\
\text{2x} \quad 1.5x \quad \text{2x} \\
\text{1.5x} \quad \text{2x} \quad \text{1.5x} \\
\end{array}
\]
<table>
<thead>
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<th>Network Congestion Game</th>
<th>Roughgarden</th>
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<td>single-commodity</td>
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Existence of Nash equilibria
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Theorem (Roughgarden and Tardos (2002))

The Nash flows of an instance are precisely the optima of a non-linear convex programming problem.

If \( f \) and \( \tilde{f} \) are Nash flows then \( \ell_e(f) = \ell_e(\tilde{f}) \) for all \( e \in E \). Hence, all Nash equilibria have the same social cost.
Existence of Nash equilibria

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**Theorem (Fabrikant et al. (2004))**

Given a network congestion game the optimal solution of the following min-cost flow problem MCF(G) yields a Nash equilibrium: For every edge \( e \in E \) we need \( N \) copies with costs \( c_{e_i} = \ell_e(i), i = 1, \ldots, N \). The capacities of all edges are 1 and we send \( N \) units of flow from \( s \) to \( t \).
Consider the following instance with $N = 2$:
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The solution with minimum social cost of 12 is given by
Consider the following instance with $N = 2$:

A Nash equilibrium with social cost of 13 is given by
Consider the following instance with $N = 2$:

A Nash equilibrium with social cost of 14 is given by
Extreme Nash equilibria

Worst Nash Equilibrium (W-NE for short):

Given: Network congestion game \((G = (V, E), (\ell_e)_{e \in E}, s \in V, t \in V, N)\) and a number \(K > 0\)

Question: Does there exist a Nash equilibrium \(f\) such that \(C_{\text{max}}(f) \geq K\)?

Best Nash Equilibrium (B-NE for short):

Given: Network congestion game \((G = (V, E), (\ell_e)_{e \in E}, s \in V, t \in V, N)\) and a number \(K > 0\)

Question: Does there exist a Nash equilibrium \(f\) such that \(C_{\text{max}}(f) \leq K\)?

Unfortunately, it can be shown that in general neither a best nor a worst Nash equilibrium is an optimal solution of \(\text{MCF}(G)\).
Theorem (Fotakis(2002), Gairing(2005))

If the users have different weights and the graph $G$ has only parallel links, $W$-NE and $B$-NE are $NP$-hard even for linear latency functions.
Nash equilibria in series-parallel graphs

The series composition $G = S(G_1, G_2)$:

Let $f_i$ be a flow in $G_i$ ($i = 1, 2$). Let $f \in f_1 \otimes f_2$ then $f$ is a Nash equilibrium in $S(G_1, G_2)$ if and only if $f_i$ are Nash equilibria in $G_i$ ($i = 1, 2$).
The parallel composition $G = S(G_1, G_2)$:

**Lemma**

Let $f_i$ be a flow in $G_i$ ($i = 1, 2$). Then $f = f_1 \cup f_2$ is a Nash equilibrium in $P(G_1, G_2)$ if and only if $f_i$ are Nash equilibria in $G_i$ ($i = 1, 2$) and $C_{\text{max}}(f_2) \leq L_{G_1}^+(f_1)$ and $C_{\text{max}}(f_1) \leq L_{G_2}^+(f_2)$. 

Network congestion games

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Worst Nash Equilibrium (W-NE for short):

Given: Network congestion game \((G = (V, E), (\ell_e)_{e \in E}, s \in V, t \in V, N)\) and a number \(K > 0\)

Question: Does there exist a Nash equilibrium \(f\) such that \(C_{\max}(f) \geq K\)?
Worst Nash equilibria in SP-graphs

Greedy Best Response (GBR):
For $i = 1$ to $N$ do
  User $i$ chooses a path with minimal latency
  with respect to load = current flow +1.
end do;
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current makespan of user 1 = 5
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current makespan of user 1 = 5
current makespan of user 2 = 6
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- current makespan of user 1 = 6
- current makespan of user 2 = 6
- current makespan of user 3 = 8
Greedy Best Response (GBR):
For $i = 1$ to $N$ do
    User $i$ chooses a path with minimal latency with respect to load = current flow +1.
end do;

The last user yields the maximum makespan!

current makespan of user 1 = 6
current makespan of user 2 = 6
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Theorem (Fotakis (2006))
Greedy Best Response yields a Nash equilibrium in series-parallel graphs.
Worst Nash equilibria in SP-graphs

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For $i = 1$ to $N$ do
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**Theorem (Fotakis (2006))**
Greedy Best Response yields a Nash equilibrium in series-parallel graphs.

**Theorem (GHKSW(2008))**
Greedy Best Response yields a worst Nash equilibrium in series-parallel graphs.
Worst Nash equilibria in arbitrary graphs

Theorem (GHKSW (2008))

Determining a worst Nash equilibrium is strongly NP-hard even for two users on acyclic networks and with linear latency functions.
Blocking Path Problem:

Given: Digraph $G = (V, E)$ with source $s \in V$ and sink $t \in V$.

Question: Does there exist an $s$-$t$-path $P \in \mathcal{P}$ such that after deleting the edges of $P$ there is no path from $s$ to $t$?
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**Theorem (GHKSW (2008))**

*The Blocking Path Problem is strongly NP-hard even on acyclic networks.*

Proof: Reduction from 3SAT.
construct positive and integral edge lengths $a_e$ such that every path from $s$ to $t$ has the same length $L^*$.
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\[
\ell_e(x) = \begin{cases} 
  a_e x & \text{if } e \in E \\
  (L^* + \frac{1}{2})(x) & \text{if } e = (s, t)
\end{cases}
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Worst Nash equilibria in arbitrary graphs

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$$\ell_e(x) = \begin{cases} a_e x & \text{if } e \in E \\ (L^* + \frac{1}{2})(x) & \text{if } e = (s, t) \end{cases}$$

$\exists$ blocking path $P^*$

$\iff$

$\exists$ Nash equilibrium $f$ for two users with $C_{\text{max}}(f) \geq L^* + \frac{1}{2}$. 
Worst Nash equilibria in arbitrary graphs

Construct positive and integral edge lengths $a_{e}$ such that every path from $s$ to $t$ has the same length $L^{*}$.

$$\ell_{e}(x) = \begin{cases} 
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## Extreme Nash Equilibria

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Best Nash Equilibrium (B-NE for short):

Given: Network congestion game \((G = (V, E), (\ell_e)_{e \in E}, s \in V, t \in V, N)\) and a number \(K > 0\)

Question: Does there exist a Nash equilibrium \(f\) such that \(C_{\max}(f) \leq K?\)
Best Nash equilibrium: $N$ is part of input

Theorem (GHKSW (2008))

Determining a best Nash equilibrium is strongly NP-hard even on series-parallel graphs and with linear latency functions if the number of users is part of the input.
Best Nash equilibrium: $N$ is part of input

Numerical 3-Dimensional Matching:

Given: Disjoint sets $X$, $Y$, $Z$, each containing $m$ elements, a weight $w(a)$ for all elements $a \in X \cup Y \cup Z$ and a bound $B \in \mathbb{Z}^+$.  

Question: Does there exist a partition of $X \cup Y \cup Z$ into $m$ disjoint sets $A_1, \ldots, A_m$ such that each $A_j$ contains exactly one element from each of $X$, $Y$ and $Z$ and $\sum_{a \in A_i} w(a) = B$ for all $i$. 
Best Nash equilibrium: $N$ is part of input

Numerical 3-Dimensional Matching:

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Assume w.l.o.g. that $w(a) \leq 2w(b)$ and $w(b) \leq 2w(a)$ for all $a, b \in X$ ($Y, Z$) holds.
Best Nash equilibrium: $N$ is part of input
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$\exists$ numerical 3-dimensional matching

$\iff$

$\exists$ Nash equilibrium $f$ for $m$ users with $C_{\text{max}}(f) \leq B$
Best Nash equilibrium: $N$ is fixed

Theorem ([GHKSW (2008)])

Determining a best Nash equilibrium is weakly NP-hard even for two users on series-parallel graphs and with linear latency functions.

Proof: Reduction from Even-Odd Partition Problem.
Best Nash Equilibrium: $N$ is fixed

A dynamic programming algorithm

Let $f$ be a Nash flow, then $C(f)$ denotes the set of latencies of the users with respect to $f$. $C(f)$ is called cost profile.
A dynamic programming algorithm

Let $f$ be a Nash flow, then $C(f)$ denotes the set of latencies of the users with respect to $f$. $C(f)$ is called cost profile.

$S_G(C)$ ... maximum latency for an additional user in a Nash flow in $G$ with cost profile $C$. 
Best Nash Equilibrium: \( N \) is fixed

A dynamic programming algorithm

Let \( f \) be a Nash flow, then \( C(f) \) denotes the set of latencies of the users with respect to \( f \). \( C(f) \) is called cost profile.

\[ S_G(C) \ldots \text{maximum latency for an additional user in a Nash flow in } G \text{ with cost profile } C. \]

Idea: Find best \( C \) such that \( S_G(C) < \infty \).
Best Nash Equilibrium: $N$ is fixed

The series composition:

$$S_G(C) = \max_{C_1 \otimes C_2 \leq C} \{S_{G_1}(C_1) + S_{G_2}(C_2)\}$$
Best Nash Equilibrium: $N$ is fixed

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The parallel composition:

$$S_G(C) = \max_{C_1 \cup C_2 = C} \min\{S_{G_1}(C_1), S_{G_2}(C_2)\}$$

$$C_1 \leq S_{G_2}(C_2)$$

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There is a huge number multisets $C$!

$O((|V| \max_{e \in N} \ell_e(N))^N)$

$\implies$ pseudopolynomial-time algorithm for fixed $N$
**Best Nash Equilibrium: \( N \) is fixed**

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\mathcal{O}\left( (|V| \max_{e \in N} \ell_e(N))^N \right)
\]

\[\implies \text{ pseudopolynomial-time algorithm for fixed } N \]

😊 Result is best possible!
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Open Questions

- Can we give a bound on the price of anarchy for the network congestion games if the graph is series-parallel?
- What can be said about the price of stability for the network congestion games if the graph is series-parallel?
Thank you for your attention!