Split Rank of Triangle and Quadrilateral Inequalities

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Joint work with Santanu Dey (CORE)
Outline

- Cuts from two rows of the simplex tableau
- The different cases to consider
- Split cuts and split ranks
- Finiteness proofs for the triangles
- The ideas for the quadrilaterals
- Conclusion
Cuts from two rows of the simplex tableau

Consider a mixed-integer program

\[ \min c^T x \]
\[ \text{s.t. } Ax = b \]
\[ x \in \mathbb{Z}^{n_1}_+ \times \mathbb{R}^{n_2}_+. \]

We consider the problem of finding valid inequalities cutting off the linear relaxation optimum.

We consider the simplex tableau

\[
\begin{align*}
    x_1 & - \bar{a}_{11}s_1 - \cdots - \bar{a}_{1n}s_n = \bar{b}_1 \\
    \vdots & \quad \vdots \\
    x_m & - \bar{a}_{m1}s_1 - \cdots - \bar{a}_{mn}s_n = \bar{b}_m.
\end{align*}
\]

- Select two rows
- Relax the integrality requirements of the non-basic variables
- Relax the nonnegativity requirements of the basic variables but keeping integrality
Cuts from two rows of the simplex tableau

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The 2 row-model

\[
\begin{pmatrix}
  x_1 \\
  x_2 
\end{pmatrix} = \begin{pmatrix}
  f_1 \\
  f_2 
\end{pmatrix} + \sum_{j=1}^{n} \begin{pmatrix}
  r_{j1} \\
  r_{j2} 
\end{pmatrix} s_j, \quad x_1, x_2 \in \mathbb{Z}, \, s_j \in \mathbb{R}_+
\]

Model studied in [Andersen, Louveaux, Weismantel, Wolsey, IPCO2007] (for the finite case) and [Cornuéjols, Margot, 2009] (for the infinite case).
The 2 row-model

The model

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\]

The geometry

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
= \begin{pmatrix}
  1/4 \\
  1/2
\end{pmatrix} + \begin{pmatrix}
  2 \\
  1
\end{pmatrix} s_1 + \begin{pmatrix}
  1 \\
  1
\end{pmatrix} s_2 + \begin{pmatrix}
  -3 \\
  2
\end{pmatrix} s_3 + \begin{pmatrix}
  0 \\
  -1
\end{pmatrix} s_4 + \begin{pmatrix}
  1 \\
  -2
\end{pmatrix} s_5
\]
The geometry

The projection picture

\[ 2s_1 + 2s_2 + 4s_3 + s_4 + \frac{12}{7}s_5 \geq 1 \]

- We project the \( n + 2 \)-dim space onto the \( x \)-space
- The facet is represented by a polygon \( L_\alpha \)
- There is no integer point in the interior of \( L_\alpha \)
- The coefficients are a ratio of distances on the figure
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\(\alpha_1, \alpha_3\)
Classification of all possible facet-defining inequalities

**Theorem**: All facets are projected to triangles and quadrilaterals [Andersen et al 2007].

Split Cut  
Triangle Cut  
Quadrilateral Cut

Cook–Kannan–Schrijver  
Disection Triangle  
Disection Quadrilateral
The split rank question

- Split cut: applying a disjunction $\pi^T x \leq \pi_0 \lor \pi^T x \geq \pi_0 + 1$ to a polyhedron $P$

  \[
  x = f + RS \\
  s_1 \geq 0 \\
  \vdots \\
  s_n \geq 0 \\
  \pi^T x \leq \pi_0
  \]

- The first split closure $P^1$ of $P$ is what you obtain after having applied all possible split disjunctions $\pi$.
- The split rank of a valid inequality is the minimum $i$ such that the inequality is valid for $P^i$.
- Most inequalities used in commercial softwares are split cuts.
- Question: what is the split rank of the 2 row-inequalities?
  In how many rounds of split cuts only can we generate the inequalities?
- The Cook-Kannan-Schrijver has infinite rank and we prove that the other triangles have finite rank.
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Useful properties of the split rank

- The split rank is invariant up to integer translation and unimodular transformation.

- (Lifting) Consider a triangle (or quadrilateral) inequality for a 3-variable problem. If we keep the same shape of the polygon and consider an $n$-variable problem, the split rank does not increase.

It allows us to work with 3 variables only when trying to find the split rank of triangles.

- (Projection) Let $\sum_{i=1}^{n} \alpha_i s_i \geq 1$ be an inequality with split rank $\eta$. Then the projected inequality $\sum_{i=1}^{n-1} \alpha_i s_i \geq 1$ has a split rank of at most $\eta$ for the projected problem.
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The triangle case

Several cases to consider, after suitable unimodular transformation

An illustration of the proof in this talk
The triangle case

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Idea of the proof of upper bounds

- We prove an upper bound on the split rank.
- Procedure: We apply a sequence of two split disjunctions.
  Successively: $x_1 \leq 0 \lor x_1 \geq 1$ and $x_2 \leq 0 \lor x_2 \geq 1$
- At step $i$, we keep one inequality of rank at most $i$ and proceed to the next disjunction.
- We prove that this procedure converges in finite time to the desired inequality.
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- We prove that this procedure **converges in finite time** to the desired inequality.
One proof for a non-degenerate non-maximal triangle

Rank 0
One proof for a non-degenerate non-maximal triangle

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Rank 1
One proof for a non-degenerate non-maximal triangle
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Rank 2
One proof for a non-degenerate non-maximal triangle

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Rank 3
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Rank 4

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Rank 5

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One proof for a non-degenerate non-maximal triangle

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The goal inequality has a rank of at most 6
The geometry behind the convergence
The geometry behind the convergence
The geometry behind the convergence
The geometry behind the convergence
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Assumptions for the following

- We have “proven” that a non-maximal triangle where the upward ray points to the left has a finite rank.
- We can prove that the constructed bound is logarithmic in the number of bits of the input.
- The proof for the upward ray pointing to the right works similarly (but not identically).
- In the following, we assume that any non-maximal triangle has a finite rank.
The maximal triangles
The maximal triangles

This inequality is a non-maximal triangle $\Rightarrow$ finite rank!
The maximal triangles
The maximal triangles

The goal inequality is valid for the disjunction.

The goal inequality is valid for the disjunction.
The maximal triangles

The goal inequality has a finite rank
The dissection triangle

Dissection $\equiv$ each side is tight at exactly one integer point
The dissection triangle

**Dissection** ≡ each side is tight at exactly one integer point

This inequality is a **non-maximal triangle** ⇒ finite rank!
The dissection triangle

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Similarly this inequality has a finite rank!
The dissection triangle

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Brown line: set of points with a *representation* that satisfy both inequalities with equality
The dissection triangle

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The dissection cut has a finite rank
The quadrilateral cuts

- Two cases: non-maximal quadrilateral and dissection quadrilateral.
- By the projection Lemma, we can deal with most non-maximal quadrilaterals.
- One exception: if the lifted triangle has infinite rank.
The quadrilateral cuts

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- By the projection Lemma, we can deal with most **non-maximal quadrilaterals**.

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- One **exception**: if the lifted triangle has **infinite rank**.
Conclusion

- All triangles except the Cook-Kannan-Schrijver have a finite rank.
- We provide a constructive split proof of that fact.
- Split cuts can essentially achieve all triangles in relatively few rounds.
- In contrast with the results of Basu et al. on the triangle closure compared to the split closure.
- Ongoing work: (almost?) all quadrilaterals have a finite rank.
- Open (and difficult) question: lower bounds on the split rank.
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