Geometric Considerations on Castel del Monte

by Susanne Krömker

Castel del Monte, a Staufer castle build during the reign of the emperor Friedrich II ($*$ 1194, † 1250) near Bari in the south of Italy, is based on the regular octagon. The important angles can be deduced from the division of a full circle by eight $360^\circ : 8 = 45^\circ$ and from the equilateral triangle which is just one eighth of an octagon $(180^\circ - 45^\circ)/2 = 67.5^\circ$.
The compass card, the so called 3-octagram, is received by lengthening the edges until they meet the straight line of the next but one edge. Small compass cards are constructed in every corner of the big compass card with the restriction that these small ones have to touch each other. The result is that their inner small octagons are the ground plans of the octagonal towers in every corner (*geometric configuration with inner aesthetics*, see [K]).

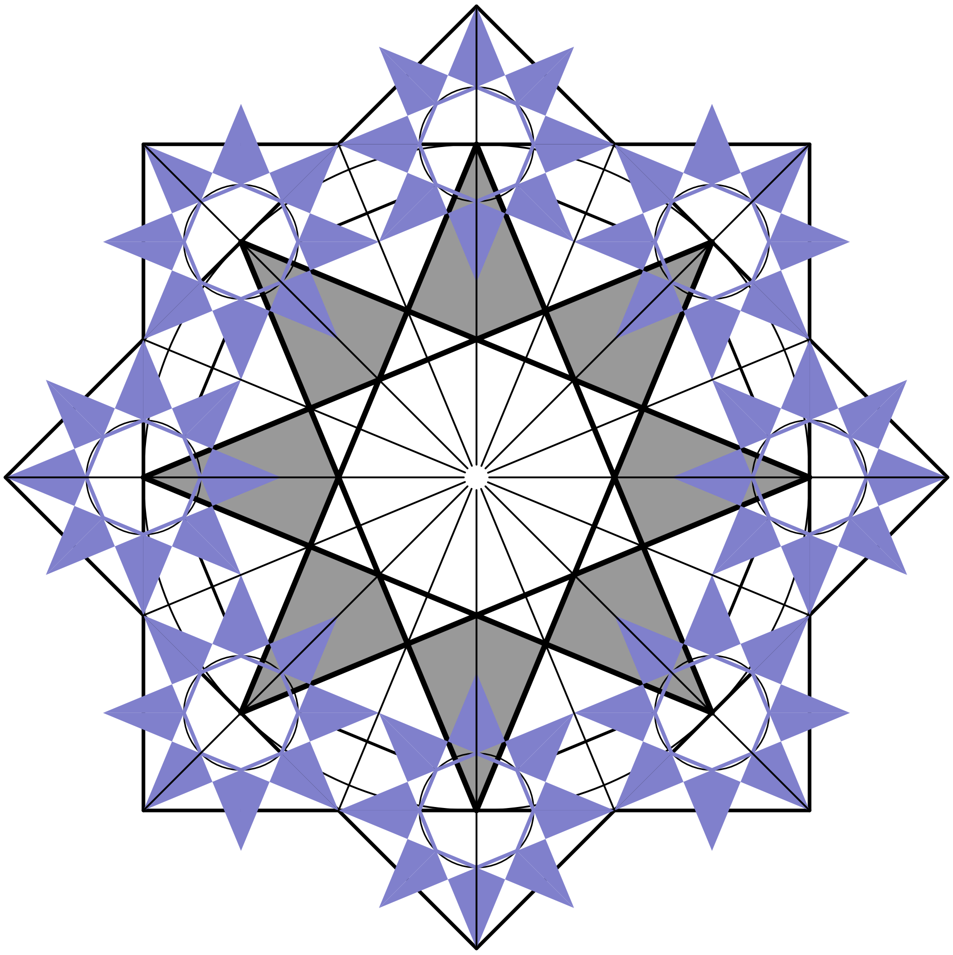

Figure 1: Construction principle.

The quotient of the radius of the circumcircle of the compass card and the circumcircle of the octagon is given by the angle function tangent (see figure 2).

$$a_0 : a_1 = \tan 67.5° = \sqrt{2} + 1 \approx 2.41421 \qquad \left(\frac{180° - 45°}{2} = 67.5°\right)$$

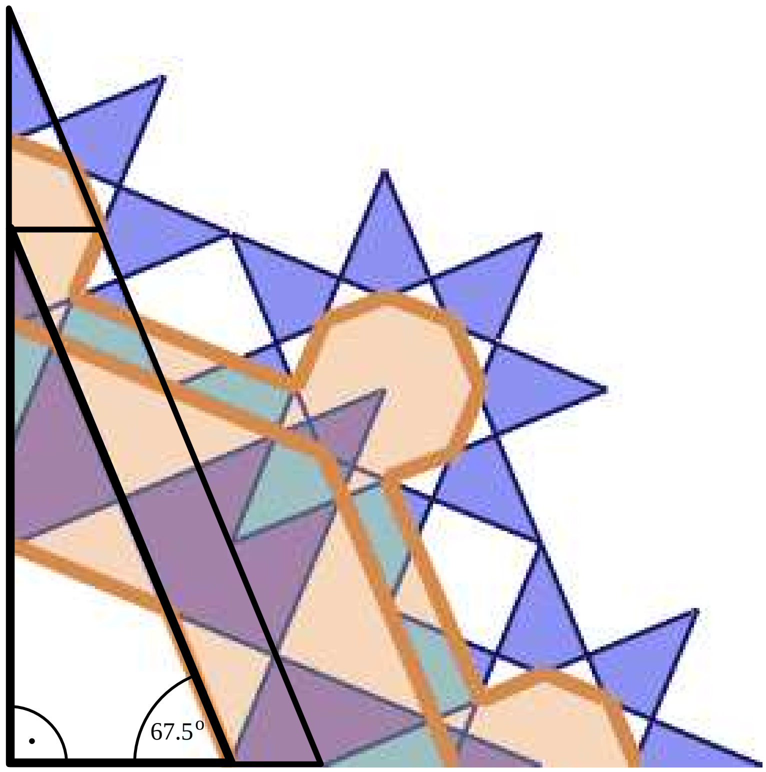

Figure 2: The outlined triangle contains the important sizes of the proportion of the building. The function of the angle tan 67.5° is exactly this proportion. By intercept theorems it is valid for each a_i of the construction principle.

If you continue this construction principle to infinity, you get a fractal. The following proportions can be deduced.

Proportions			Distances
a_0	$=$	$\sqrt{2}+1$	from the center of the inner yard to the center of a tower
a_1	$=$	1	from the center of the inner yard to one of its corners (unity)
a_2	$=$	$\sqrt{2}-1$	from the center of a tower to one of its corners
$\dots$			
a_n	$=$	$\frac{a_{n-1}}{a_{n-2}}$	from the center of the n^{th} tower to one of its corners

Taking the formula for a_n it follows by induction that the radius of the n^{th} tower is given by $a_n = a_0 \left(\frac{1}{a_0^n}\right)$. The ray from the center of the building through a corner of the compass card is composed of a sequence of a_i that determines an ultimate radius of the construction principle. This is because the sum $\sum_{k=0}^{\infty} a_k$ converges (see figure 3).

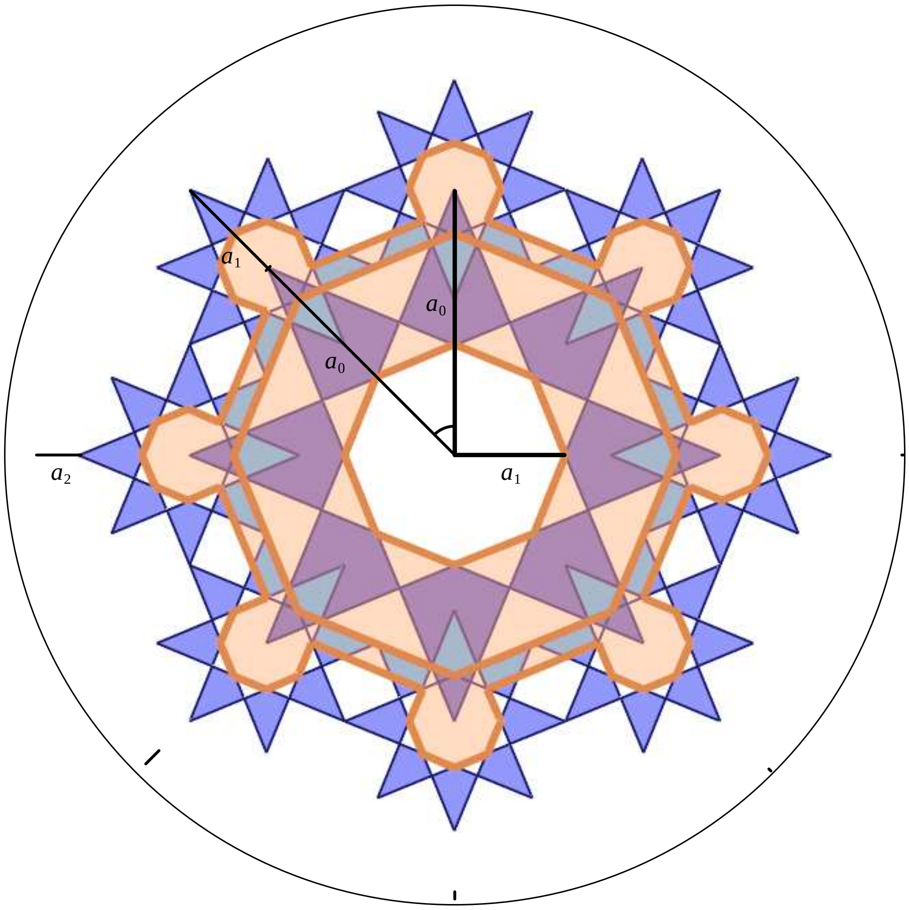

Figure 3: Ground plan with the sizes of a_0, a_1 and a_2. The outer circle marks the convergence radius.

$$\sum_{k=0}^{\infty} a_k = a_0 \sum_{k=0}^{\infty} \frac{1}{a_0^k} = a_0 \left(\frac{1}{1-\frac{1}{a_0}}\right) = \frac{a_0^2}{a_0 - 1} = \frac{(\sqrt{2}+1)^2}{\sqrt{2}} = 2 + \frac{3}{\sqrt{2}} \approx 4.12132$$

The distance from the center of the inner yard to one of its corners is chosen to be the unity $a_1 = 1$. The ultimate spatial extension of the playful construction principle is 4.12132 times the unity.

Following H. Götze ([G], p. 164) the length of a_0 is 23.70 m. The unity a_1 therefore equals 9.8169 m.
The height of the building in the computer generated pictures is assumed to be twice the unity a_1. Thereby each tower is as high as its distance from its most inner point to the center of the building (see figure 4).

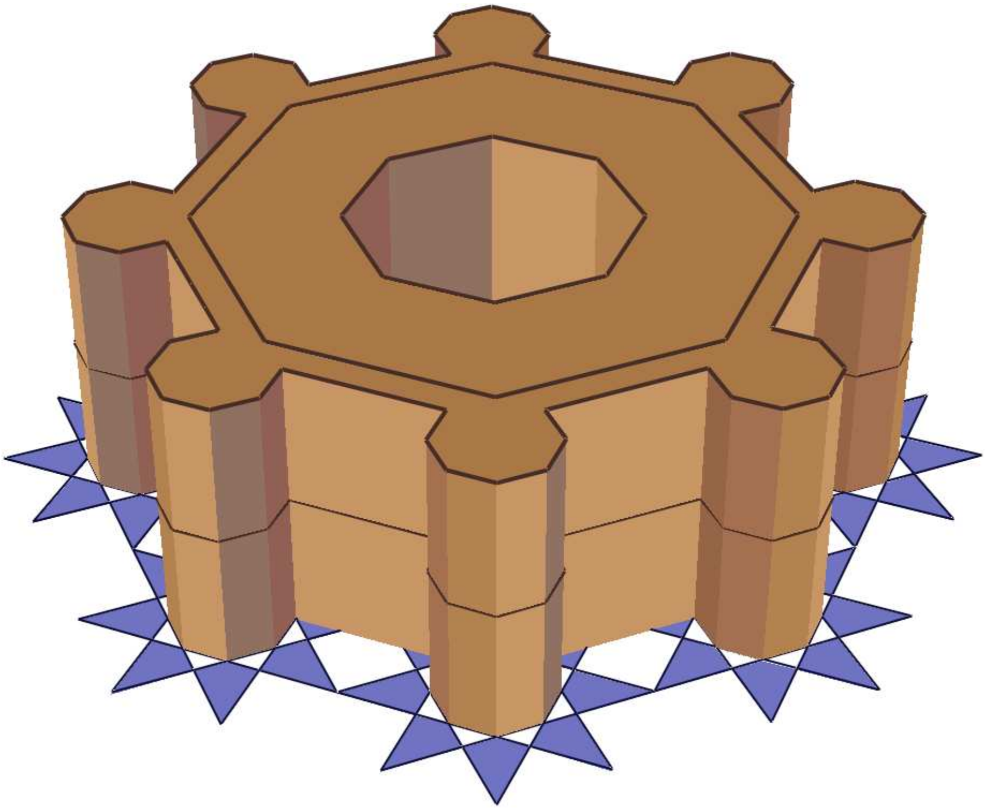

Figure 4: Ground plan with construction principle and perspective elevation of the Castel with a height of twice the unity a_1.

References

[G] H. Götze,
Castel del Monte, Gestalt und Symbol der Architektur Friedrichs II.
3. Auflage, Prestel, München 1991

[K] M. Koecher,
Castel del Monte und das Oktogon, Miscellanea mathematica, Springer, Heidelberg 1991