

Analytical study of the effects of some soil and plant parameters on root growth due to absorption of one mobile ion: A free-boundary model

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Received 10 July 1992: Accepted in revised form 16 June 1993

Key words: balance integral method, free boundary problem, quasi-stationary method, root growth, root-soil interface.

Abstract

The object of our study is : A model for root growth through a free boundary problem and the effects resulting from differences in nutrient availability and transport of one only mobile nutrient between the root surface and the rhizosphere produced by a absorption Michaelis-Menten for low and high concentrations. The model equations are solved by two methods: the quasi-stationary method and the balance integral method. The numerical solutions are used to compute radial root growth. Curves of nutrient concentration at the root-soil interface curves as a function of root radius as well as curves representing root radius as a function of time are plotted. The parameters which are varied are the root absorption power, flux velocity at the root surface, efflux, rhizosphere radius, diffusion coefficient, buffer power, and maximum influx. The two methods show the theoretical results for radial root growth in the range of low and high concentrations. The balance integral method provides more detailed information .

Introduction

One of the methods for studying the nutrient uptake by plant roots, which can be a satisfactory method of modelling the plant-root system is by use of the partial differential equation for convective and diffusive flow to a root, and simulation models for nutrient uptake have frequently been used in the evaluation of the effect of soil and root characteristics on nutrient uptake (Nye et al., 1969; Claasen et al., 1976; Cushman, 1982). These models have only considered nutrient uptake and have supposed an exponential root growth, without taking into account effects of coupling between nutrient transport and root growth. Because of the difficulties in modelling these complex processes, in recent articles (Reginato et al., 1990, 1991), as a first approximation to a real, more complete description, a method for computing the effects on root growth owing to absorption of one nearly immobile nutrient by means of a free boundary problem has been proposed. In these articles, the root radius $s(t)$ as a function of time and the interface concentration $C(s)$ as a function of the root radius $s(t)$ are computed by the quasi-stationary method (Crank, 1984) with a contour condition representing a fixed rhizosphere border. Also, the interface concentration $C(s(t),t)$ as an explicit function of $s(t)$ and the time t , as a the root radius $s(t)$ vs. t are computed by the balance integral method (Goodman, 1958; Reginato et al. 1992). Both methods are based in the principle of conservation of mass in the soil as well as at the root-soil interface. The goal of the present paper is to compute the root radius $s(t)$ as a function of t and the interface concentration $C(s(t),t)$ as a function of $s(t)$ through the quasi-stationary and the balance integral methods with

a null flux condition on rhizosphere border valid for mobile ions, in order to estimate the effects of different parameters on root growth for low and high concentrations (not saline conditions).

The free boundary model and governing equations

Because of the rather complicated scenario of the plant-root system, as a first step toward achieving some understanding of the physical and chemical processes involved, an idealized one-dimension diffusion-convection ion transport and radial root growth model was chosen for this study. As described for us (Reginato et al., 1990, 1991) it is assumed that : a vertical cylindrical root is immersed in a porous homogeneous and isotropic medium (the soil) while moisture, temperature and light conditions are maintained at a steady state. It is assumed that the nutrient uptake occurs at the root absorption zone, and the root hairs are not considered in the present model. The rate of uptake can be described by a Michaelis Menten equation, and the rate of influx at infinite concentration (J_m) and the Michaelis Menten constant (K_m) are independent of the flux velocity of soil solution at the root (v_o). The nutrient transport occurs via convection and diffusion in the radial direction only (the latter takes place in soil solution phase only). It is assumed that the diffusion coefficient (D) and the buffer power (b) are independent of the nutrient concentration, which implies that there is a linear relation between C and C_i , where C is the total diffusable ion concentration and C_i is the ion concentration in soil solution. No allowance for a change in J_m , K_m , or E (efflux) with age is made. It is also assumed that the coefficient diffusion is independent of the flux, and the convective velocity of water at root surface is not affected by nutrient concentration. Production or depletion of nutrient by microbial or other activity is considered null, and owing to the proposed model not taking into account the energy balance implicit in the root metabolism, we assume that the nutrient taken up is totally available for root growth. At this point, only a fraction is available for root growth and the remaining nutrient is available for shoot growth. So, we arrive to only qualitative conclusions. From now on, we shall denote C_i by C for convenience in the notation. The governing equations for convective and diffusive transport of nutrient to the root (Cushman, 1982) as well as the governing equations for root growth at the root-soil interface for low concentrations ($J_m \sim kC$) are given in the following free boundary problem to one phase (the soil) (Crank, 1984; Tarzia, 1988; Tarzia et al. 1989) (In order to simplify the model, without loss of generality, radial growth in cylindrical coordinates is considered) by:

$$\begin{aligned}
 \text{i)} \quad & DC_{rr} + D(1 + \epsilon)\frac{C_r}{r} = 0, & s(t) < r < R, 0 < t < T \\
 \text{ii)} \quad & C(r,0) = \Phi(r), & s_o \leq r \leq R \\
 \text{iii)} \quad & -DbC_r(R,t) + v_oC(R,t) = 0, & 0 < t < T \\
 \text{iv)} \quad & DbC_r(s(t),t) + v_oC(s(t),t) = kC(s(t),t) - E = aC(s(t),t)v_s(t), & 0 < t < T \\
 \text{v)} \quad & s(0) = s_o, & 0 < s_o < R
 \end{aligned} \tag{1}$$

where: $C_r = \frac{\partial C}{\partial r}$, $C_{rr} = \frac{\partial^2 C}{\partial r^2}$, $C_t = \frac{\partial C}{\partial t}$, r is the position coordinate, t the time, T is the time for which the solution exists. Equation i) represents a simple application of the principle of conservation of mass (in soil) under steady moisture conditions with the nutrient flux consisting of two components (diffusive and convective) (Cushman, 1982), Equation ii) is the initial condition, and Equation iii) is the boundary condition on the rhizosphere radius taking into account inter-root competition for the nutrient considered, and is a more realistic condition when the more mobile nutrients are considered, Equation iv) representing the mass nutrient balance conditions at the soil-root interface, and equation v) is the initial condition for the free boundary $s(t)$. $v_s(t) = \frac{ds(t)}{dt}$ is the rate of radial root growth, a is a stoichiometric coefficient, R is the rhizosphere radius, and s_o is the initial root radius. The parameter ϵ is given by $\epsilon = \frac{v_o s_o}{Db} > 0$. $\Phi(r)$ is the initial concentration profile (given by the equation (11) below). The two free boundary conditions (1-iv) can be written by :

$$C_r(s(t),t) = g(C(s(t),t)), \quad t > 0 \tag{2}$$

$$v_s(t) = f(C(s(t),t)), \quad t > 0 \tag{3}$$

where functions g and f are given by:

$$g(C) = \frac{1}{Db} [(k - v_o)C - E] \tag{4}$$

$$f(C) = \frac{1}{a} \left[k - \frac{E}{C} \right] \tag{5}$$

which satisfy the following properties:

$$f(C) > 0 \Leftrightarrow C > C_p = \frac{E}{k} \tag{6}$$

$$g(C) > 0 \Leftrightarrow C > C_m = \frac{E}{(k-v_0)}, \quad (C_m > C_p) \quad (7)$$

The solution to problem (1) is given by:

$$C(r,t) = \beta(t) - \frac{\alpha(t)}{r^\epsilon}, \quad s(t) < r < R, \quad t > 0 \quad (8)$$

where:

$$\alpha(t) = \left[\frac{E}{Db} \right] \frac{1}{\frac{(k-v_0)}{Db} \left[\frac{1}{R^\epsilon} \alpha_0 - \frac{1}{s(t)^\epsilon} \right] - \frac{\epsilon}{s(t)^{\epsilon+1}}} = \alpha_2 \frac{s(t)^{\epsilon+1}}{\beta_1 I\left(\frac{s(t)}{R}\right) - 1} \quad (9)$$

$$\beta(t) = \frac{\alpha(t)}{R^\epsilon} \left[1 + \frac{s_0}{R} \right] = \alpha_0 \frac{\alpha(t)}{R^\epsilon} = \alpha_0 \frac{\alpha_2}{\beta_1 I\left(\frac{s(t)}{R}\right) - 1} \frac{s(t)^{\epsilon+1}}{R^\epsilon} \quad (10)$$

$$\begin{aligned} \Phi(r) &= E \frac{\left[\frac{1}{R^\epsilon} \left(1 + \frac{s_0}{R} \right) - \frac{1}{r^\epsilon} \right]}{(k-v_0) \left[\frac{1}{R^\epsilon} \left(1 + \frac{s_0}{R} \right) \right] - \frac{k}{s_0}} = \alpha(0) \left[\frac{\alpha_0}{R} - \frac{1}{r^\epsilon} \right] = \\ &= \frac{\beta_2}{\beta_1 I\left(\frac{s_0}{R}\right) - 1} \left[\frac{s_0}{r} \right]^{\epsilon+1} I\left(\frac{r}{R}\right) \end{aligned} \quad (11)$$

and $s(t)$ is the unique solution of the following Cauchy problem (Ince, 1956; Kreider et al., 1968):

$$\begin{aligned} v_s(t) &= \frac{v_0}{a} L\left(\frac{s(t)}{R}\right), \quad t > 0 \\ s(0) &= s_0 \in (0, R) \end{aligned} \quad (12)$$

where:

$$\alpha_2 = \frac{E}{v_0 s_0}, \quad \beta_1 = \frac{R(k-v_0)}{v_0 s_0} > 0, \quad \beta_2 = \frac{RE}{v_0 s_0} > 0, \quad \alpha_0 = 1 + \frac{s_0}{R} \quad (13)$$

$$I(x) = x(\alpha_0 x^\epsilon - 1), \quad L(x) = 1 + \frac{s_0}{R} \frac{1}{I(x)} \quad \text{with } x \in (0, 1) \quad (14)$$

Moreover, the interface concentration is given by the following expression:

$$C(s(t), t) = \frac{\beta_2 I\left(\frac{s(t)}{R}\right)}{\beta_1 I\left(\frac{s(t)}{R}\right) - 1}, \quad t > 0 \quad (15)$$

that is, the interface concentration does not depend explicitly on variable t .

We can remark that the solution, given above, for the problem (1) exists and has a physical meaning if the conditions

$$x_3(w) < \xi < 1, \quad \xi = \frac{s_0}{R} \in (0, 1), \quad (16)$$

$$\frac{k}{v_0} > \psi\left(\rho(\xi)\right) = \frac{\rho\left(\frac{s_0}{R}\right)}{\rho\left(\frac{s_0}{R}\right) - 1} = \frac{(1+\xi)\xi^\epsilon}{(1+\xi)\xi^{\epsilon-1}}, \quad \epsilon = \theta\xi, \quad \theta = \frac{Rv_0}{Db} > 0 \quad (17)$$

among parameters $\frac{k}{v_0}$, $\frac{s_0}{R}$, θ are satisfied (See Appendix A for more details), and $x_3 = x_3(\theta) > 0$ is the unique solution of the equation:

$$\rho(x) = 1, \quad x \in (0, 1) \quad (18)$$

with:

$$\rho(x) = (1+x)x^{\theta x} \quad \text{with } x \in (0, 1), \quad (19)$$

$$\psi(x) = \frac{x}{x-1} \quad \text{with } x \in (1, 2). \quad (20)$$

The solution of the Cauchy problem (12) is computed numerically by the Runge-Kutta method for ordinary differential equations (Conte and Boor, 1972). Figures 1 to 12 represent theoretical results for the interface concentration $C(s(t), t)$ vs. s and the root radius $s(t)$ vs. t as a function of absorption power k for low concentrations and parameters which satisfy the condition (17). For simplicity we choose the parameter a equals to 1, i.e. we assume that for each ion which arrives to root only one carrier is available for the absorption. Results of a sensitivity analysis using the free boundary model are shown in Figure 3; the initial values used for the analysis are shown in the same figure. Each parameter has varied between 0.5 and 2.0 of its initial value, whilst all other parameters were held constant at initial values following a similar methodology given in Barber, 1984.

A similar set of equations for high concentrations are given in the following free boundary problem assuming that the maximum influx is given by $J \sim J_m$ ($J_m - E > 0$) if we replace condition (1-iv) by :

$$\text{iv)} \quad \text{Db}C_r(s(t), t) + v_o C(s(t), t) = J_m - E = aC(s(t), t) v_s(t), \quad 0 \leq t \leq T \quad (21)$$

Now, $f(C)$ and $g(C)$ are given by:

$$g(C) = \frac{1}{\text{Db}} [J_m - E - v_o C] \quad (22)$$

$$f(C) = \frac{J_m - E}{a} \left[\frac{1}{C} \right] \quad (23)$$

which satisfy the following properties:

$$g(C) > 0 \Leftrightarrow C < C_q = \frac{J_m - E}{v_o} \quad (24)$$

$$f(C) > 0, \quad \forall C > 0 \quad (25)$$

The solution of problem for high concentrations is given by :

$$C(r, t) = \gamma(t) - \frac{\delta(t)}{r^\epsilon}, \quad s(t) < r < R, \quad t > 0 \quad (26)$$

where :

$$\delta(t) = \left[\frac{J_m - E}{\text{Db}} \right] \frac{1}{\frac{v_o}{\text{Db}} \left[\frac{1}{R^\epsilon} \left(1 + \frac{s_0}{R} \right) - \frac{1}{s(t)^\epsilon} \right] + \frac{\epsilon}{s(t)^{\epsilon+1}}} = \frac{J_m - E}{v_o} \frac{R^\epsilon}{\alpha_o + \left(\frac{R}{s(t)} \right)^\epsilon \left(\frac{s_0}{s(t)} - 1 \right)} \quad (27)$$

$$\gamma(t) = \frac{\delta(t)}{R^\epsilon} \left[1 + \frac{s_0}{R} \right] = \alpha_o \frac{\delta(t)}{R^\epsilon} \quad (28)$$

$$\Phi(r) = \frac{(J_m - E)}{v_o} \left[1 - \left(\frac{R}{r} \right)^\epsilon \frac{1}{\alpha_o} \right] \quad (29)$$

and $s(t)$ is the unique solution of the following Cauchy problem:

$$\frac{ds(t)}{dt} = \frac{v_o}{a} L \left(\frac{s(t)}{R} \right), \quad s(0) = s_0 \in (0, R), \quad t > 0 \quad (30)$$

where I and L are defined before. Function $\Phi = \Phi(r)$ satisfies the following conditions

$$\Phi(0^+) = -\infty, \quad \Phi(R_1) = \frac{(J_m - E)}{v_o} \frac{\xi}{1 + \xi} > 0, \quad \Phi' > 0 \text{ in } (0, R).$$

Moreover, the coefficient $\xi = \frac{s_0}{R}$ must verifies the condition:

$$\Phi(s_0) = \frac{(J_m - E)}{v_o} \left[1 - \frac{1}{\rho(\xi)} \right] > 0$$

which is equivalent to the condition $x_3(\theta) < \xi = \frac{s_0}{R} < 1$, where $x_3 = x_3(\theta)$ and function ρ are defined before. Therefore, we obtain, that the interface concentration is given by the following expression:

$$C(s(t),t) = \frac{(J_m - E)}{v_o} \frac{1}{L\left(\frac{s(t)}{R}\right)} = \frac{(J_m - E)}{v_o} \frac{1}{v_s(t)}, \quad t > 0 \quad (31)$$

Moreover, we see that $v_s(t) > 0$, $\forall t > 0$ because $\xi = \frac{s_0}{R} > x_o = \frac{1}{(1+\xi)^{\theta\zeta}}$ is obviously verified.

The solution of Cauchy problem (30) is computed numerically by the Runge-Kutta method for ordinary differential equations. Figures 4 to 5 represent theoretical results for the interface concentration $C(s(t),t)$ vs. s and the interface position $s(t)$ vs. t for high concentrations as a function of rhizosphere radius R . Results of a sensitivity analysis using the quasi-stationary method for high concentrations are shown in Figure 6.

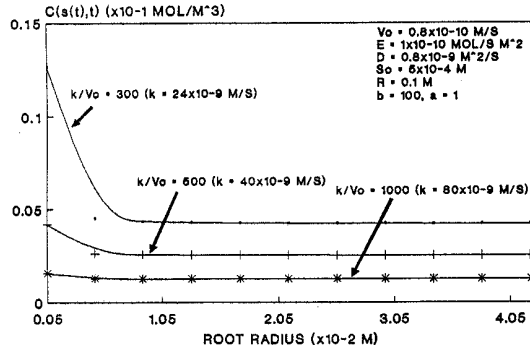


Fig. 1. Interface concentration $C(s(t), t)$ vs. s as a function of absorption power k for low ion concentrations (quasi-stationary method).

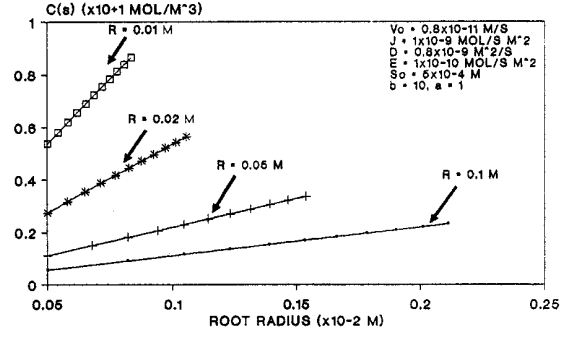


Fig. 4. Interface concentrations $C(s(t), t)$ vs. s as a function of radius R for high ion concentrations (quasi-stationary method).

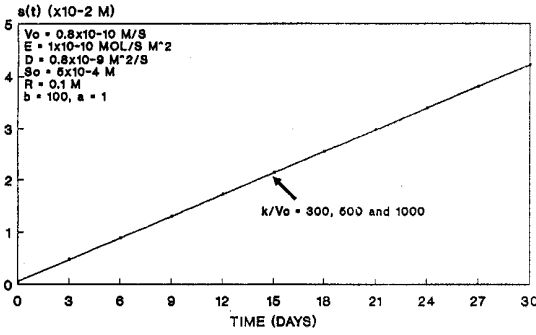


Fig. 2. Root radius $s(t)$ vs. t as a function of absorption power k for low ion concentrations (quasi-stationary method).

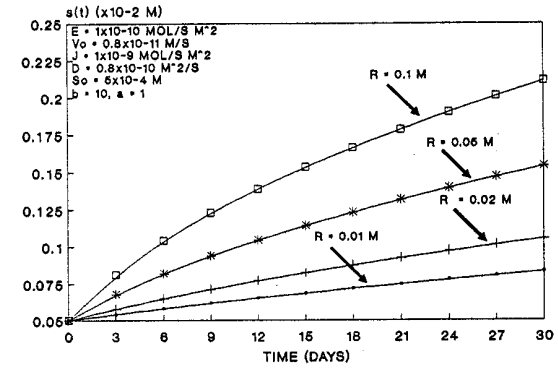


Fig. 5. Root radius $s(t)$ vs. t as a function of radius R for high ion concentrations (quasi-stationary method).

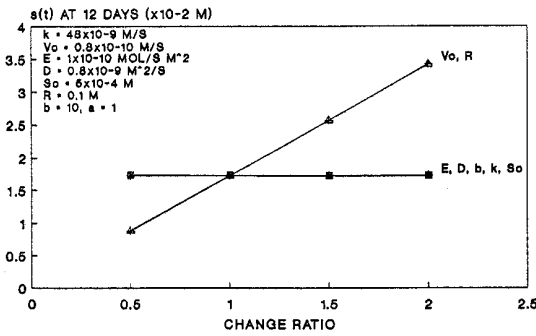


Fig. 3. Results of a sensitivity analysis of parameters for $s(t)$ for low ion concentrations through the quasi-stationary method.

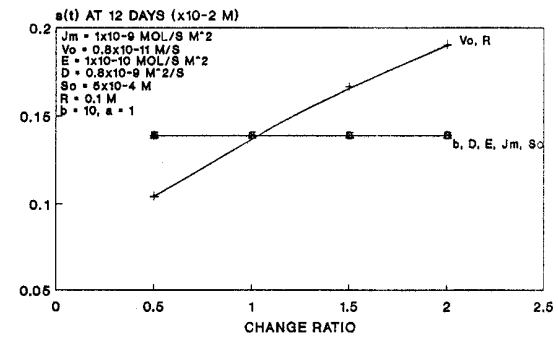


Fig. 6. Results of a sensitivity analysis of parameters for $s(t)$ for high ion concentrations through quasi-stationary method.

In order to compute $C(s(t), t)$ as a explicit function of $s(t)$ and t , and $s(t)$ for a more general rank of concentrations we propose the following free boundary problem with the conditions 1-iv) given by:

$$\text{iv) } DbC_r(s(t), t) + v_0 C(s(t), t) = \frac{kC(s(t), t)}{1 + \frac{kC(s(t), t)}{J_m}} - E = aC(s(t), t) v_s(t), \quad 0 < t < T \quad (32)$$

The two free boundary conditions $f(C)$ and $g(C)$ can be now written by:

$$g(C) = \frac{1}{Db} \left[\frac{kC}{1 + \frac{k}{J_m} C} - E - v_o C \right] \quad (33)$$

$$f(C) = \frac{1}{a} \left[\frac{k}{1 + \frac{k}{J_m} C} - \frac{E}{C} \right] \quad (34)$$

which satisfy the following properties:

$$g(C) > 0 \Leftrightarrow C_m^- = \frac{-J_m + \delta^2(J_m - E) - d}{2k} < C < C_m^+ = \frac{-J_m + \delta^2(J_m - E) + d}{2k} \quad (35)$$

where $d = \sqrt{[-J_m + \delta^2(J_m - E)]^2 - 4\delta^2 J_m E}$ with the hypothesis: $\delta^2 = \frac{k}{v_o} > 1$, $\frac{E}{J_m} < \left(1 - \frac{1}{\delta^2}\right)^2$, and

$$f(C) > 0 \Leftrightarrow C > C_p = \frac{E}{k \left[1 - \frac{E}{J_m}\right]} \quad \text{with the hypothesis } \frac{E}{J_m} < 1 \quad (36)$$

The interval (C_m^-, C_m^+) represents the range of concentrations for which $g(C) > 0$ and C_p represent the minimum soil solution concentration required for root growth.

To solve the problem with (32) (that is, to compute $C = C(r, t)$ (in particular, $C = C(s(t), t)$ and the free boundary interface $r = s(t)$ a priori unknown) we apply the mass balance integral method (Goodman, 1958; Reginato and Tarzia, 1992) to the present case for root growth. The solution is found integrating the partial differential equation (1-i) in r on the domain $(s(t), R)$. Thus:

$$\int_{s(t)}^R C_t(r, t) dr = D \int_{s(t)}^R C_{rr}(r, t) dr + D(1 + \epsilon) \int_{s(t)}^R \frac{C_r(r, t)}{r} dr \quad (37)$$

and we propose:

$$C(r, t) = \Phi(r) [1 + \eta(t)(R - r)^2] \quad (38)$$

which depend on the parameters of the system and satisfy the initial and boundary conditions 1-ii) and 1-iii), that is:

$$C(r, 0) = \Phi(r) \Leftrightarrow \eta(0) = 0, \quad (39)$$

$$-DbC_r(R, t) + v_o C(R, t) = 0 \Leftrightarrow -Db\Phi'(R) + v_o \Phi(R) = 0. \quad (40)$$

We denote $\nu = \nu(t)$ by:

$$\nu(t) = C(s(t), t) = \Phi(s(t)) [1 + \eta(t)(R - s(t))^2] \quad (41)$$

which depends on the parameters of the system through $s(t)$, $\Phi(s(t))$ and $\eta(t)$. Replacing (40) and (41) in Eq. (37), after some elementary manipulations, the problem is reduced to :

$$\int_{s(t)}^R C_t(r, t) dr = D [C_r(R, t) - g(\nu(t))] + D(1 + \epsilon) \left[\frac{C(R, t)}{R} - \frac{\nu(t)}{s(t)} + \int_{s(t)}^R \frac{C_r(r, t)}{r^2} dr \right] \quad (42)$$

$$v_s(t) = f(\nu(t)), \quad s(0) = s_o \quad t > 0$$

Replacing (38) in Eq. (42), after some elementary manipulations we obtain the following system of two coupled ordinary differential equations (see Appendix B.) (valid for the cases $\epsilon \neq 1, 2, 3$):

$$\begin{aligned} \frac{d\eta(t)}{dt} &= \frac{F_1 + F_2 + D(1 + \epsilon)(F_3 + F_4 + F_5 + F_6 + F_7 + F_8)}{(F_9 + F_{10} + F_{11})} \eta(0) \\ &= 0 \\ \frac{ds(t)}{dt} &= \frac{1}{a} \left[\frac{k}{1 + \frac{k}{J_m} \Phi(s(t)) [1 + \eta(t)(R - s(t))^2]} - \frac{E}{\Phi(s(t)) [1 + \eta(t)(R - s(t))^2]} \right] = f(\nu(t)), \\ s(0) &= s_o \end{aligned} \quad (43)$$

and:

$$\Phi(r) = BA - \frac{A}{r^\epsilon} \quad \text{with } B = \frac{1}{R^\epsilon} \left[1 + \frac{s_0}{R} \right], \quad A = \frac{E}{(k - v_0)B - \frac{k}{s_0}} \quad (44)$$

which verifies conditions (39) and (40)

The initial profile concentration $\Phi(r)$ given by the Eq. (44) above has been computed by the quasi-stationary method (Reginato et al., 1990) for low concentrations and it is determined by the system. The functions F_i are given by:

$$F_1 = \left[\frac{v_0}{b} + \frac{D(1+\epsilon)}{R} \right] \frac{As_0}{R^{\epsilon+1}}, \quad F_3 = - \frac{\Phi(s(t)) [1 + \eta(t)(R - s(t))^2]}{s(t)}$$

$$F_2 = - \frac{1}{b} \left[\frac{k\Phi(s(t))[1 + \eta(t)(R - s(t))^2]}{1 + \frac{k}{j_m}\Phi(s(t))[1 + \eta(t)(R - s(t))^2]} - E - v_0\Phi(s(t))[1 + \eta(t)(R - s(t))^2] \right] = - Dg(\nu(t))$$

$$F_4 = AB [1 + \eta(t)R^2] \left[\frac{1}{s(t)} - \frac{1}{R} \right], \quad F_5 = - 2AB\eta(t)R \ln \left[\frac{R}{s(t)} \right] + AB\eta(t) [R - s(t)]$$

$$F_6 = - \frac{A[1 + \eta(t)R^2]}{(\epsilon+1)} \left[\frac{1}{s^{\epsilon+1}(t)} - \frac{1}{R^{\epsilon+1}} \right], \quad F_7 = \frac{2}{\epsilon} A\eta(t)R \left[\frac{1}{s^\epsilon(t)} - \frac{1}{R^\epsilon} \right]$$

$$F_8 = \frac{A\eta(t)}{(1-\epsilon)} \left[\frac{1}{s^{\epsilon-1}(t)} - \frac{1}{R^{\epsilon-1}} \right], \quad F_9 = AB R^2 [R - s(t)] - \frac{AR^2}{(1-\epsilon)} [R^{(1-\epsilon)} - s^{(1-\epsilon)}(t)]$$

$$F_{10} = - AB R [R^2 - s^2(t)] + \frac{2AR}{(2-\epsilon)} [R^{(2-\epsilon)} - s^{(2-\epsilon)}(t)], \quad F_{11} = \frac{AB}{3} [R^3 - s^3(t)] - \frac{A}{(3-\epsilon)} [R^{(3-\epsilon)} - s^{(3-\epsilon)}(t)]$$

We can remark that for the particular cases $\epsilon = 1, 2$ and 3 , a similar system to (43) of two ordinary differential equations can be obtained. The solution to system (43) is computed numerically by the Runge-Kutta method for a system of ordinary differential equations. Figures 7 and 8 represent theoretical results for the interface concentration $C(s(t), t)$ vs. s and the interface position $s(t)$ vs. t as a function of absorption power k in the range of low concentrations and parameters which satisfy the condition (17).

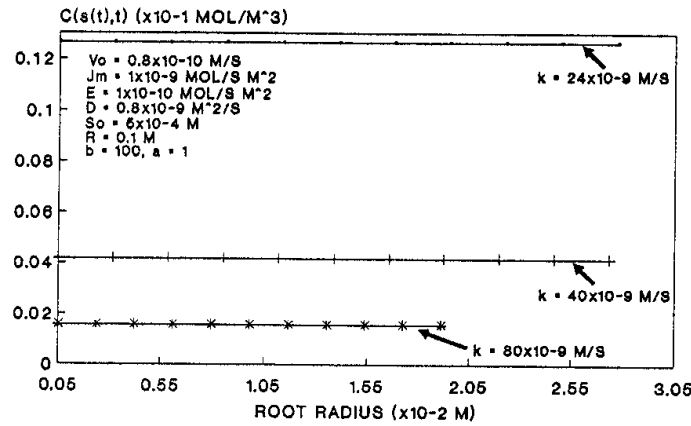


Fig. 7. Interface concentration $C(s(t), t)$ vs. s as a function of absorption power k for low ion concentrations through integral balance.

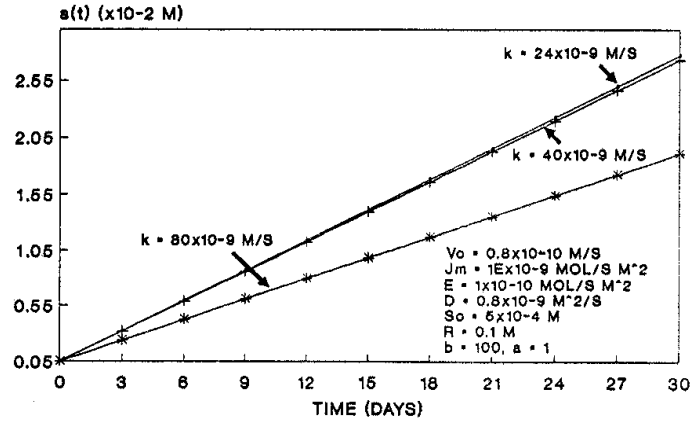


Fig. 8. Root radius $s(t)$ vs. t as a function of absorption power k for low ion concentrations (integral balance).

Results of a sensitivity analysis using the balance integral method for low concentrations are shown in Figure 9.

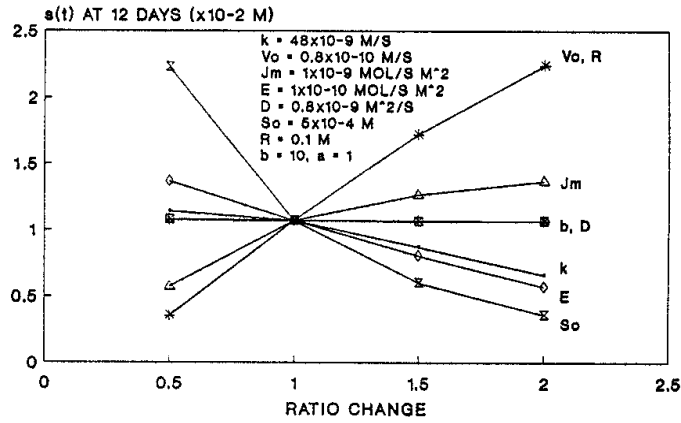


Fig. 9. Results of a sensitivity analysis of parameters for $s(t)$ through the balance integral method for low ion concentrations.

For high concentrations we obtain a quite similar system of ordinary differential equations with the initial profile of concentrations (computed by the quasistationary method for high concentrations) given by:

$$\Phi(r) = BA - \frac{A}{r^c} \quad (45)$$

where :

$$B = \frac{1}{R^c} \left(1 + \frac{s_0}{R} \right), \quad A = \frac{J_m - E}{\frac{v_0}{R^c} \left(1 + \frac{s_0}{R} \right)} = \frac{J_m - E}{B v_0}$$

Figures 10 and 11 represent theoretical results for $C(s(t), t)$ vs. s and $s(t)$ vs. t respectively as a function of rhizosphere radius R in the range of high concentrations and Figure 12 represent the results of sensitivity analysis for predicted root growth through the balance integral method for high concentrations.

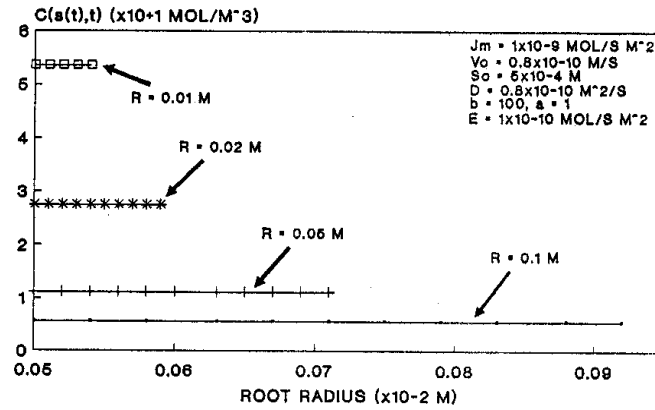


Fig. 10. Interface concentration $C(s(t), t)$ vs. s as a function of parameter R for high ion concentrations (integral balance).

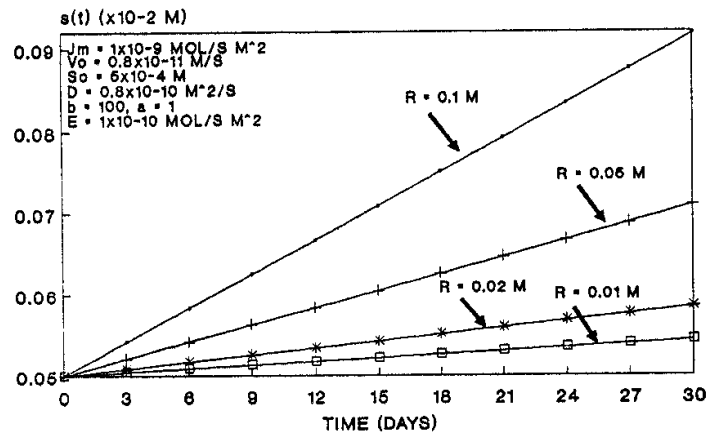


Fig. 11. Root radius $s(t)$ vs. t as a function of radius R for high ion concentrations (integral balance).

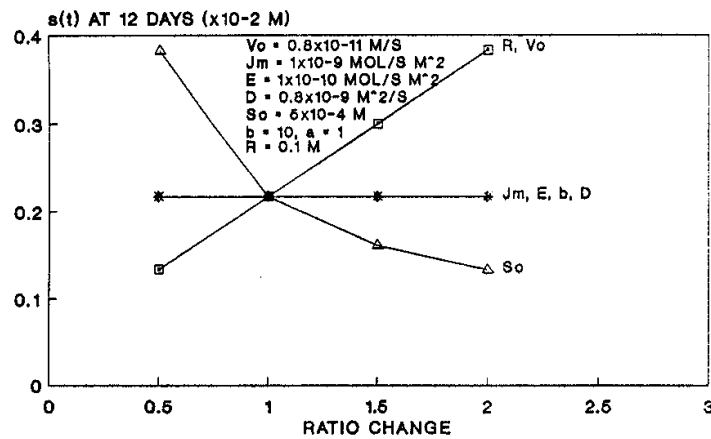


Fig. 12. Results of sensitivity analysis of parameters for $s(t)$ through the balance integral method for high ion concentrations.

Conclusions

From the analysis of the results given by the quasi-stationary method, we conclude that for low concentrations the results of sensitivity analysis indicate that predicted $s(t)$ increases rapidly with an increase

in v_o and R (Figure 3). Values for k , b , D , E , and s_o had little or no effect because the mathematical approximation of method is poor. On the other hand, for low concentrations the results of sensitivity analysis obtained for the balance integral method (Figure 9) likewise indicates that predicted $s(t)$ increases rapidly with v_o and R . The increase in v_o represents an increase in total nutrient available for the plant, since, if the remaining parameters are held constants, increasing v_o increases the convective flux to root. Increasing R increases the rhizosphere volume and so a greater ion quantity is available. At the same time we conclude that the predicted $s(t)$ increases moderately with J_m . Values of b and D had little or no effect, whilst increasing k , E , and s_o the predicted $s(t)$ decreases. The decrease with k (Figure 8) occurs because the gradient at root-soil interface (given by the Eq. (4)) as a function of k and $C(s)$ decreases (the expression $g(C(s(t),t) = \frac{1}{Db} [(k - v_o)C(s(t),t) - E]$ takes into account the combined effects of increase in k and the decrease of $C(s(t),t)$ as a function of k (Figure 7)). The decrease with s_o occurs because, although the root surface increases as s_o^2 , the rhizosphere volume available decreases as $L(R^2 - s_o^2)$ where L is the length root and taking into account that $R \gg s_o$ then $(R^2 - s_o^2) > s_o^2$. (No allowance for a change in L with time is made because we have only considered radial growth).

From the analysis of the results obtained by the quasi-stationary method, the interface concentration become nearly constant after a week elapsed (Compare Figs.1 and 2), whilst from the results obtained by the balance integral method the interface concentration is constant throughout the period simulated (Figure 7) in agreement with the De Willigen's results (De Willigen, 1981) on N uptake. From the analysis of the results obtained by the quasi-stationary method and the balance integral method for high concentrations, the results of sensitivity analysis indicate that predicted $s(t)$ increases rapidly with an increase in v_o and R (Figures 6 and 12). Values for b , D , E and J_m were without effect or had little effect. The predicted $s(t)$ decrease with respect to s_o .

By comparing the results obtained by the quasi-stationary method and the balance integral method we can conclude that: firstly, the mathematics of the quasi-stationary method allow analytical expressions which must be satisfied by the system parameters, such as the inequality (17) and, secondly the general qualitative behavior of the results is similar for both methods, although the balance integral method gives us a more detailed theoretical information with respect to dependence on system parameters, for example, for low concentrations, the variation of $s(t)$ vs. t with respect to the parameter k is negligible for the quasi-stationary method (Figure 3) whilst the balance integral method gives us an appreciable change (Figure 9).

In conclusion, this paper represents a qualitative approach for the effects on root growth due to absorption of only one mobile nutrient. From comparison between the results of the present model and the model for ions that are relatively immobile for low and high concentrations (Reginato et al. 1990, 1991) we conclude that the efficiency of absorption kinetics ($J = kC$ or $J = J_m$, respectively) is greater for low concentrations for both models. Moreover, these models are useful as a basis for developing more complex models for transport of nutrients and their effect on root growth, as for example, these same models taking into account the root length (the effects of some parameters could be change) and simultaneous absorption of one or more immobile and mobile ions.

Acknowledgements

This paper has been sponsored by the Projects "Problemas de Frontera Libre de la Física-Matemática" from CONICET, Rosario (Argentina) and "Problemas de Frontera Libre y Crecimiento de Raíces de Cultivos" from UNRC and CONICOR, Río Cuarto (Argentina). The authors wish to thank the anonymous referee for his helpful suggestions.

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Appendix

Part A (*quasi-stationary method*): The expression for $C(r,t)$ and $s(t)$ can be obtained by a similar method developed in Reginato et al. (1990). Function I , given by (14), satisfies the following properties:

$$\begin{aligned} I(0) &= 0, & I(x_0) &= 0, & I(1) &= \frac{s_0}{R} \\ I &< 0 \text{ in } (0, x_0), & I &> 0 \text{ in } (x_0, 1), & I' &> 0 \text{ in } [x_0, 1] \\ I'(x_1) &= 0, & I(x_1) &= -\frac{\epsilon}{\epsilon+1} x_1 \\ I' &< 0 \text{ in } (0, x_1), & I' &> 0 \text{ in } (x_1, 1) \\ \epsilon \leq I'(x) \leq \mu &= \epsilon + (1 + \epsilon) \frac{s_0}{R} \text{ in } (x_0, 1), \end{aligned}$$

where:

$$x_0 = \frac{1}{(\alpha_0)^{1/\epsilon}}, \quad \alpha_0 = 1 + \frac{s_0}{R}, \quad \epsilon = \frac{v_0 s_0}{D_b} = \theta \frac{s_0}{R}, \quad 0 < x_1 = \frac{1}{((1+\epsilon)\alpha_0)^{1/\epsilon}} < x_0$$

Owing to $(\xi = \frac{s_0}{R} \in (0, 1))$:

$$\beta_1 I\left(\frac{s(t)}{R}\right) - 1 = \frac{k-v_0}{v_0} \frac{1}{\xi} I\left(\frac{s(t)}{R}\right) - 1 \geq \frac{k-v_0}{v_0} \frac{I(\xi)}{\xi} - 1 > 0, \quad \forall t > 0$$

the condition (17) is a consequence from the following equivalences:

$$\begin{aligned} \beta_1 I\left(\frac{s(t)}{R}\right) - 1 > 0, \forall t > 0 &\Leftrightarrow \frac{k}{k-v_0} < \rho(\xi) < 2 \text{ with } \xi \in (0, 1) \Leftrightarrow \\ &\Leftrightarrow \frac{k}{v_0} > \frac{\rho(\xi)}{\rho(\xi)-1} = \psi(\rho(\xi)) \text{ with } \rho(\xi) \in (1, 2) \Leftrightarrow \\ &\Leftrightarrow \frac{k}{v_0} > \psi(\rho(\xi)) \text{ with } x_3(\theta) < \xi < 1 \end{aligned}$$

where $x_3 = x_3(\theta) > 0$ is the unique solution of the equation $\rho(x) = 1$ with $x \in (0, 1)$.

Moreover, function $\rho = \rho(x)$ verifies the following conditions:

$$\begin{aligned} \rho(0^+) &= 1, \quad \rho(1) = 2 \\ \rho' &> 0 \text{ in } (x_4(\theta), 1), & \rho' &< 0 \text{ in } (0, x_4(\theta)) \\ \exists! x_4 &= x_4(\theta) \in (0, 1) & \text{such that } \rho'(x_4) &= 0, & \rho''(x_4) &> 0 \\ \exists! x_3 &= x_3(\theta) \in (x_4, 1) & \text{such that } \rho(x_3) &= 1 \end{aligned}$$

Then the solution of the ordinary differential equation (12) is well defined because

$$|L'(x)| \leq \frac{(k-v_0)^2}{v_0^2 s_0} R \mu, \quad \forall x \in [x_0, 1]$$

Part B (*Integral balance*): Replacing (38) in Eq.(42), after some manipulations, we obtain:

$$\begin{aligned} \int_{s(t)}^R C_t(r,t) dr &= \int_{s(t)}^R \Phi(r) \dot{\eta}(t) [R_1 - r]^2 dr = \\ &= R^2 \dot{\eta}(t) \int_{s(t)}^R \Phi(r) dr - 2R \dot{\eta}(t) \int_{s(t)}^R r \Phi(r) dr + \dot{\eta}(t) \int_{s(t)}^R \Phi(r) r^2 dr \end{aligned}$$

and, taking into account (44), we obtain:

$$\begin{aligned}
\int_{s(t)}^R \Phi(r) \, dr &= AB \int_{s(t)}^R dr - A \int_{s(t)}^R r^{-\epsilon} dr = \\
&= AB [R - s(t)] - \frac{A}{(1-\epsilon)} [R^{(1-\epsilon)} - s^{(1-\epsilon)}(t)] \text{ (with } \epsilon \neq 1)
\end{aligned}$$

$$\begin{aligned}
\int_{s(t)}^R r \Phi(r) \, dr &= AB \int_{s(t)}^R r \, dr - A \int_{s(t)}^R r^{(1-\epsilon)} dr \\
&= \frac{AB}{2} [R^2 - s^2(t)] - \frac{A}{(2-\epsilon)} [R^{(2-\epsilon)} - s^{(2-\epsilon)}(t)] , \text{ (with } \epsilon \neq 2)
\end{aligned}$$

$$\begin{aligned}
\int_{s(t)}^R r^2 \Phi(r) \, dr &= AB \int_{s(t)}^R r^2 \, dr - A \int_{s(t)}^R r^{(2-\epsilon)} dr \\
&= \frac{AB}{3} [R^3 - s^3(t)] - \frac{A}{(3-\epsilon)} [R^{(3-\epsilon)} - s^{(3-\epsilon)}(t)] , \text{ (with } \epsilon \neq 3)
\end{aligned}$$

then:

$$\begin{aligned}
\int_{s(t)}^R C_t(r,t) \, dr &= R^2 \dot{\eta}(t) AB [R - s(t)] - \frac{AR^2 \dot{\eta}(t)}{(1-\epsilon)} [R^{(1-\epsilon)} - s^{(1-\epsilon)}(t)] - \\
&- 2R \dot{\eta}(t) \frac{AB}{2} [R^2 - s^2(t)] + \frac{A2R \dot{\eta}(t)}{(2-\epsilon)} [R^{(2-\epsilon)} - s^{(2-\epsilon)}(t)] + \\
&+ \dot{\eta}(t) \frac{AB}{3} [R^3 - s^3(t)] - \frac{A \dot{\eta}(t)}{(3-\epsilon)} [R^{(3-\epsilon)} - s^{(3-\epsilon)}(t)] . \tag{B1}
\end{aligned}$$

Similarly, we obtain:

$$\begin{aligned}
\int_{s(t)}^R \frac{C(r,t)}{r^2} \, dr &= \int_{s(t)}^R \frac{1}{r^2} \left[AB - \frac{A}{r^\epsilon} \right] [1 + \eta(t)(R - r)^2] \, dr = \\
&= AB [1 + \eta(t)R^2] \int_{s(t)}^R \frac{dr}{r^2} - [2AB\eta(t)R] \int_{s(t)}^R \frac{dr}{r} + AB\eta(t) \int_{s(t)}^R dr - \\
&- A [1 + \eta(t)R^2] \int_{s(t)}^R \frac{dr}{r^{\epsilon+2}} + 2A\eta(t)R \int_{s(t)}^R \frac{dr}{r^{\epsilon+1}} - A\eta(t) \int_{s(t)}^R \frac{dr}{r^\epsilon} \\
\int_{s(t)}^R \frac{C(r,t)}{r^2} \, dr &= AB [1 + \eta(t)R^2] \left[\frac{1}{s(t)} - \frac{1}{R} \right] - 2AB\eta(t) R \ln \frac{R}{s(t)} - AB\eta(t) [R - s(t)] - \\
&- \frac{A [1 + \eta(t)R^2]}{(1+\epsilon)} \left[\frac{1}{s^{(\epsilon+1)}(t)} - \frac{1}{R^{(\epsilon+1)}} \right] - \frac{2A\eta(t)R}{\epsilon} \left[\frac{1}{s^\epsilon(t)} - \frac{1}{R^\epsilon} \right] - \\
&- \frac{A\eta(t)}{(1-\epsilon)} \left[\frac{1}{R^{(\epsilon-1)}} - \frac{1}{s^{(\epsilon-1)}(t)} \right] \tag{B2}
\end{aligned}$$

Finally, replacing (B1) and (B2) in Eq. (42), after elementary manipulations, we obtain the system (43).

Nomenclature

r :	Position (radius) coordinate	[cm]
t :	Time	[Sec]
J_m :	Rate of influx at infinite concentration	$\left[\frac{\text{Mol}}{\text{Sec-Cm}^2} \right]$
K_m :	Michaelis Menten constant	$\left[\frac{\text{Mol}}{\text{Cm}^3} \right]$
k :	Absorption power of root	$\left[\frac{\text{Cm}}{\text{Sec}} \right]$
v_o :	Flux velocity of soil solution at root surface	$\left[\frac{\text{Cm}}{\text{Sec}} \right]$
D :	Effective diffusion coefficient	$\left[\frac{\text{Cm}^2}{\text{Sec}} \right]$
C :	Total diffusible ion concentration	$\left[\frac{\text{Mol}}{\text{Cm}^3} \right]$
C_i :	Ion concentration in soil solution	$\left[\frac{\text{Mol}}{\text{Cm}^3} \right]$
$C(s(t),t)$:	Ion concentration in soil solution at root-soil interface	$\left[\frac{\text{Mol}}{\text{Cm}^3} \right]$
$\Phi(r)$:	Initial soil solution concentration profile	$\left[\frac{\text{Mol}}{\text{Cm}^3} \right]$
s_o :	Initial radius	[Cm]
E :	Constant efflux	$\left[\frac{\text{Mol}}{\text{Sec-Cm}^2} \right]$
$s(t)$:	Instantaneous root radius	[Cm]
$v_s(t)$:	Rate of radial root growth	$\left[\frac{\text{Cm}}{\text{Sec}} \right]$
L :	Root lenght	[cm]
R :	Rhizosphere radius	[Cm]
b :	Buffer power	Dimensionless
a :	Stoichiometric coefficient	Dimensionless
ϵ	Parameter	Dimensionless
α_1	Parameter	$\left[\frac{1}{\text{Cm}} \right]$
α_2	Parameter	$\left[\frac{\text{Mol}}{\text{Cm}^4} \right]$
$\alpha(t), \delta(t)$	Coefficients	$\left[\frac{\text{Mol}}{\text{Cm}^2} \right]$
$\beta(t), \gamma(t), \nu(t)$	Coefficients	$\left[\frac{\text{Mol}}{\text{Cm}^3} \right]$
$\eta(t)$	Coefficient	$\left[\frac{1}{\text{Cm}^2} \right]$

